High-Rate Sparse Superposition Codes with Iteratively Optimal Estimates

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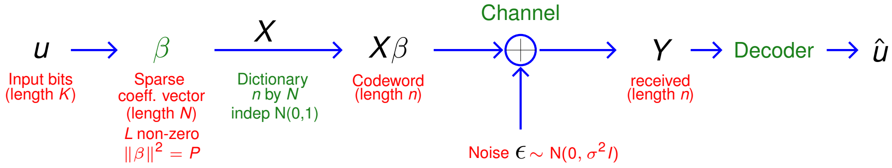
Sparse Superposition Code for the Gaussian Channel

$$u \longrightarrow \beta \xrightarrow{\quad X \quad} X\beta \longrightarrow \overset{\text{Channel}}{\oplus} \longrightarrow Y \longrightarrow \text{Decoder} \longrightarrow \hat{u}$$

- u: Input bits (length K)
- β: Sparse coeff. vector (length N), L non-zero, $\|\beta\|^2 = P$
- X: Dictionary n by N indep N(0,1)
- $X\beta$: Codeword (length n)
- Channel: Noise $\epsilon \sim$ N(0, $\sigma^2 I$)
- Y: received (length n)

Sparse Superposition Code for the Gaussian Channel



Linear Model $\quad Y = X\beta + \epsilon$

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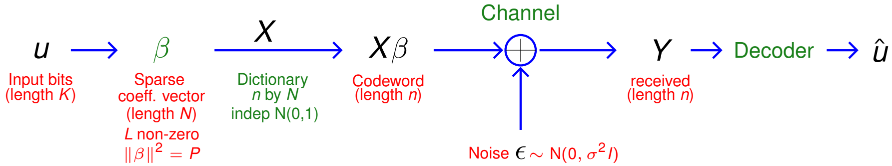

- Partitioned Coef.: $\beta = (00{*}0000,\ 000{*}000, \ldots, 0{*}00000)$
- L sections of size $M = N/L$, one non-zero in each

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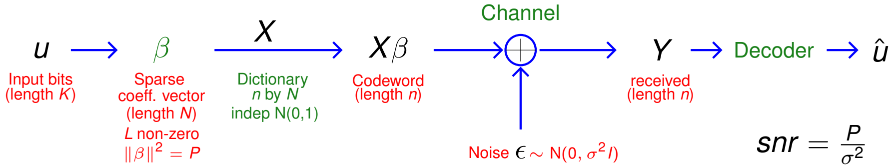

$$snr = \frac{P}{\sigma^2}$$

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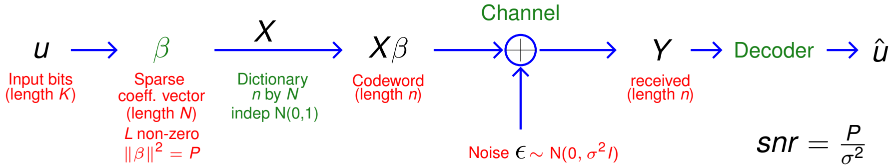

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- Ultra-sparse case: Impractical $M = 2^{nR/L}$ with L constant
  (successive decoder reliable for $R < C$: Cover 1972 IT)
- Moderately-sparse: $M = L^a$ with $n = (L \log M)/R$

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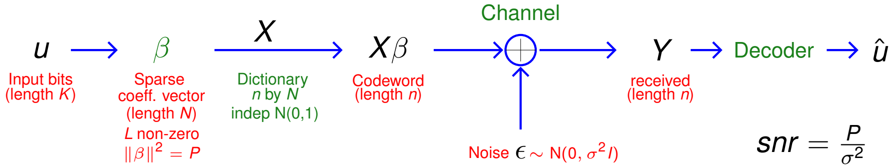

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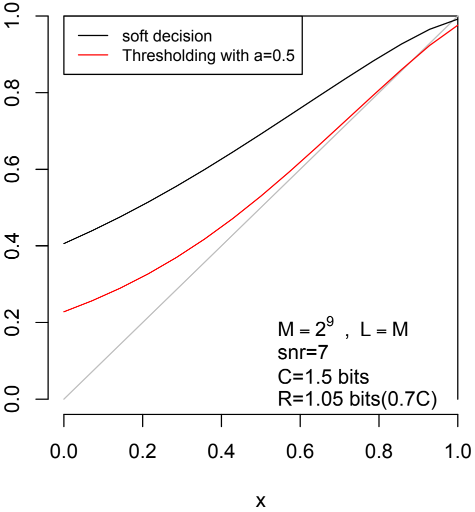
  (reliable for $R < C$)

  Maximum likelihood decoder (Joseph & Barron 2010a ISIT, 2012a IT)

  Adaptive successive decoder with threshold (J&B 2010b ISIT, 2012b)

  Adaptive successive decoder with soft decision (B&C, this talk)

Progression of success rate



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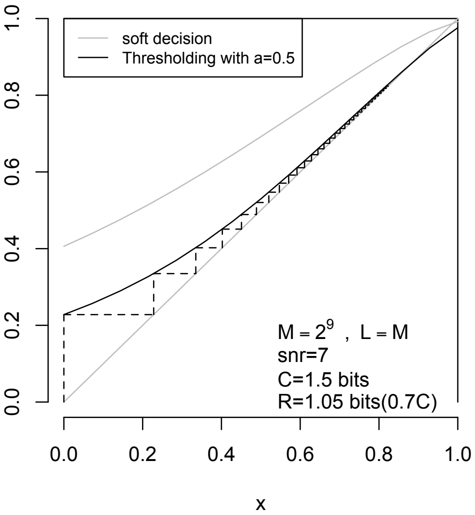

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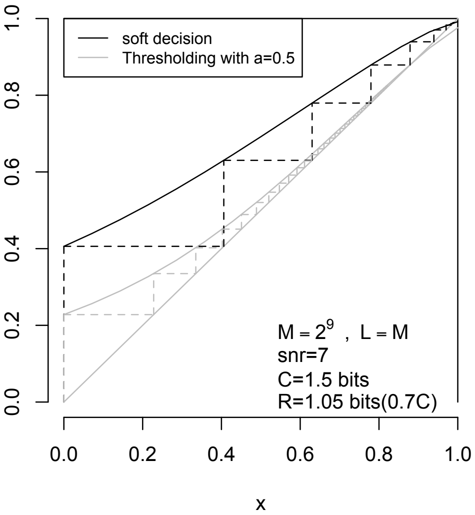

Power Allocation

- Power control: $\sum_{\ell=1}^{L} P_\ell = P \qquad \|\beta\|^2 = P$
- Special choice: P_ℓ proportional to $e^{-2C\ell/L}$ for $\ell = 1, \ldots, L$

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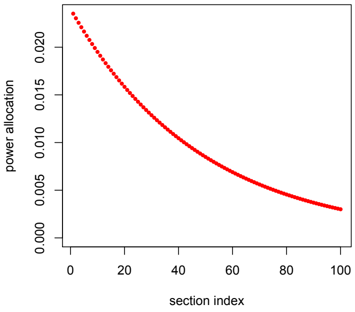

Coefficient vectors β

- Power control: $\sum_{\ell=1}^{L} P_\ell = P \qquad\qquad \|\beta\|^2 = P$
- Special choice: P_ℓ proportional to $e^{-2C\ell/L}$ for $\ell = 1, \ldots, L$
- Coeff. sent: $\beta \; = (00\sqrt{P_1}0000, 000\sqrt{P_2}000, \ldots, 0\sqrt{P_L}00000)$
- Terms sent: $(j_1, j_2, \ldots, j_L)$
- $\beta_j = \sqrt{P_\ell}\,1_{\{j=j_\ell\}}$ for j in section ℓ , for $\ell = 1, \ldots, L$
- $\mathcal{B} =$ set of such allowed vectors β for codewords $X\beta$

Coefficient Estimates $\hat{\beta}$

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- $\mathcal{B} =$ set of such allowed vectors β for codewords $X\beta$
- $\hat{\beta}_j$ restricted to $\mathcal{B}$ or the convex hull of $\mathcal{B}$
- $\hat{\beta}_j = \sqrt{P_\ell}\, \hat{w}_j$ for j in sec_ℓ , with $\hat{w}_j \geq 0$, $\sum_{j \in sec_\ell} \hat{w}_j = 1$

Iterative Estimation

For $k \geq 1$

- Coefficient fits: $\hat{\beta}_{k,j}$ (initially 0)
- Codeword fits: $F_k = X\hat{\beta}_k$
- Vector of statistics: $stat_k =$ function of $(X, Y, F_1, \ldots, F_k)$
- e.g. $stat_{k,j}$ proportional to $X_j^T(Y - F_k)$
- Update $\hat{\beta}_{k+1}$ as a function of $stat_k$

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  - Thresholding: Adaptive Successive Decoder
    $\hat{\beta}_{k+1,j} = \sqrt{P_\ell}$ if $stat_{k,j}$ is above threshold in sections ℓ not previously decoded

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- Coefficient fits: $\hat{\beta}_{k,j}$ (initially 0)
- Codeword fits: $F_k = X\hat{\beta}_k$ also $F_{k,-j} = X\hat{\beta}_{k,-j}$
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- e.g. $stat_{k,j}$ proportional to $X_j^T(Y - F_{k,-j})$
- Update $\hat{\beta}_{k+1}$ as a function of $stat_k$
  - Thresholding: Adaptive Successive Decoder

    $\hat{\beta}_{k+1,j} = \sqrt{P_\ell}$ if $stat_{k,j}$ is above threshold in sections ℓ not previously decoded
  - Soft decision:

    $\hat{\beta}_{k+1,j} = \mathbb{E}[\beta_j | stat_k]$

    with thresholding on the last step

Statistics

- $stat_k$ = function of $(X, Y, F_1, \ldots, F_k)$ $\qquad F_k = X\hat{\beta}_k$
- Orthogonalization : Let $G_0 = Y$ and for $k \geq 1$

$$G_k = \text{part of } F_k \text{ orthogonal to } G_0, G_1, \ldots, G_{k-1}$$

- Components of statistics

$$\mathcal{Z}_{k,j} = \frac{X_j^T G_k}{\|G_k\|}$$

- Class of statistics $stat_k$ formed by combining $\mathcal{Z}_0, \ldots, \mathcal{Z}_k$

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- Class of statistics $stat_k$ formed by combining $\mathcal{Z}_0, \ldots, \mathcal{Z}_k$

$$stat_{k,j} = \mathcal{Z}_{k,j}^{comb} + \frac{\sqrt{n}}{\sqrt{c_k}}\,\hat{\beta}_{k,j}$$

where $\mathcal{Z}_k^{comb} = \sqrt{\lambda_{k,0}}\,\mathcal{Z}_0 - \sqrt{\lambda_{k,1}}\,\mathcal{Z}_1 - \ldots - \sqrt{\lambda_{k,k}}\,\mathcal{Z}_k$

with $\quad \lambda_{k,0} + \lambda_{k,1} + \ldots + \lambda_{k,k} = 1$

Statistics based on residuals

Let $stat_{k,j}$ be proportional to $X_j^T(Y - X\hat{\beta}_{k,-j})$

$$stat_{k,j} = \frac{X_j^T(Y - X\hat{\beta}_k)}{\sqrt{nc_k}} + \frac{\|X_j\|^2}{\sqrt{nc_k}}\,\hat{\beta}_{k,j}$$

Arises with $\underline{\lambda}_k$ proportional to

$$\Big(\,(\|Y\| - \mathcal{Z}_0^T\hat{\beta}_k)^2,\ (\mathcal{Z}_1^T\hat{\beta}_k)^2, \ldots, (\mathcal{Z}_k^T\hat{\beta}_k)^2\,\Big)$$

and $nc_k = \|Y - X\hat{\beta}_k\|^2$.

Here, c_k is typically between σ^2 and $\sigma^2 + P$

Idealized Statistics

$\underline{\lambda}_k$ exists yielding $stat_k^{ideal}$ with distributional representation

$$\frac{\sqrt{n}}{\sqrt{\sigma^2 + \|\beta - \hat{\beta}_k\|^2}}\beta \; + \; Z_k^{comb}$$

with $\qquad Z_k^{comb} \sim N(0, I)$.

This is a normal shift that improves with decreasing $\|\beta - \hat{\beta}_k\|^2$.

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For terms sent the shift $\alpha_{\ell,k}$ has an effective *snr* interpretation

$$\alpha_{\ell,k} \;=\; \sqrt{n\,\frac{P_\ell}{\sigma^2 + P_{remaining,k}}}$$

where $\quad P_{remaining,k} = \|\beta - \hat{\beta}_k\|^2$.

Distributional Analysis

Lemma 1: shifted normal conditional distribution

Given $\mathcal{F}_{k-1} = (\|G_0\|, \ldots, \|G_{k-1}\|, \mathcal{Z}_0, \mathcal{Z}_1, \ldots, \mathcal{Z}_{k-1})$, the $\mathcal{Z}_k$ has the distributional representation

$$\mathcal{Z}_k = \frac{\|G_k\|}{\sigma_k} b_k + Z_k$$

- $\|G_k\|^2/\sigma_k^2 \sim \text{Chi-square}(n-k)$
- $Z_k \sim N(0, \Sigma_k)$ indep of $\|G_k\|$

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- $b_0, b_1, \ldots, b_k$ the successive orthonormal components of

$$\begin{bmatrix} \beta \\ \sigma \end{bmatrix}, \begin{bmatrix} \hat{\beta}_1 \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} \hat{\beta}_k \\ 0 \end{bmatrix} \qquad (*)$$

- $\Sigma_k = I - b_0 b_0^T - b_1 b_1^T - \ldots - b_k b_k^T$
  $= \text{projection onto space orthogonal to } (*)$
- $\sigma_k^2 = \hat{\beta}_k^T \Sigma_{k-1} \hat{\beta}_k$

Distribution of $\mathcal{Z}_k^T = \left(\frac{X_1^T G_k}{\|G_k\|}, \ldots, \frac{X_N^T G_k}{\|G_k\|}, \frac{\epsilon^T G_k}{\sigma\|G_k\|}\right)$

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Idealized Statistics

Weights of combination based on $\underline{\lambda}_k$ proportional to

$$\Big((\sigma_Y - b_0^T \hat{\beta}_k)^2,\ (b_1^T \hat{\beta}_k)^2, \ldots, (b_k^T \hat{\beta}_k)^2 \Big)$$

produces the desired distributional representation

$$stat_k^{ideal} = \frac{\sqrt{n}}{\sqrt{\sigma^2 + \|\beta - \hat{\beta}_k\|^2}}\,\beta + Z_k^{comb}$$

with $Z_k^{comb} \sim N(0, I)$ and $\sigma_Y^2 = \sigma^2 + P$.

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with $Z_k^{comb} \sim N(0, I)$ and $\sigma_Y^2 = \sigma^2 + P$.

- $\|\beta - \hat{\beta}_k\|^2$ is close to its known expectation
- This provides approximation of the distribution of the $stat_{k,j}$ as independent shifted normals.

Relationship between statistics

The stats based on residuals estimate the idealized statistics.

Why? For $stat_k^{ideal}$ the $\underline{\lambda}_k$ are proportional to

$$n\left((\sigma_Y - b_0^T\hat{\beta}_k)^2,\ (b_1^T\hat{\beta}_k)^2, \ldots, (b_k^T\hat{\beta}_k)^2\right)$$

whereas, for the residual-based $stat_k$ they are proportional to

$$\left((\|Y\| - \mathcal{Z}_0^T\hat{\beta}_k)^2,\ (\mathcal{Z}_1^T\hat{\beta}_k)^2, \ldots, (\mathcal{Z}_k^T\hat{\beta}_k)^2\right)$$

Here $\mathcal{Z}_{k'}^T\hat{\beta}_k/\sqrt{n}$ is approximately $b_{k'}^T\hat{\beta}_k$ for $k' \leq k$.

Indeed, with the chi-square factor replaced by its expectation,

$$\mathcal{Z}_{k'}^T\hat{\beta}_k/\sqrt{n} = b_{k'}^T\hat{\beta}_k + Z_{k'}^T\hat{\beta}_k/\sqrt{n}.$$

The $Z_{k'}^T\hat{\beta}_k$ has mean 0 and is stochastically dominated by $Z_{k'}^T\beta$.

Iteratively Bayes optimal coefficient estimates

With prior $j_\ell \sim \mathit{Unif}$ on sec_ℓ, the Bayes estimate based on stat_k

$$\hat{\beta}_{k+1} = \mathbb{E}[\beta | \mathit{stat}_k]$$

has representation $\hat{\beta}_{k+1,j} = \sqrt{P_\ell}\, \hat{w}_{k,j}$ with

$$\hat{w}_{k,j} = \mathit{Prob}\{j_\ell = j | \mathit{stat}_k\}.$$

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Here, when the $stat_{k,j}$ are independent $N(\alpha_{\ell,k} 1_{\{j=j_\ell\}}, 1)$, we have the logit representation

$$\hat{w}_{k,j} = \frac{e^{\alpha_{\ell,k}\, stat_{k,j}}}{\sum_{j \in sec_\ell} e^{\alpha_{\ell,k}\, stat_{k,j}}}.$$

In our setting, $\alpha_{\ell,k}$ is the shift given by

$$\alpha_{\ell,k} = \sqrt{\frac{n\, P_\ell}{\sigma^2 + \mathbb{E}\|\beta - \hat{\beta}_k\|^2}}$$

Relating error rate and squared distance

- Error of posterior weight is $(1 - \hat{w}_{k,j_\ell})$ if j_ℓ is sent.
- The power-weighted error

$$\sum_{\ell=1}^{L} P_\ell \, (1 - \hat{w}_{k,j_\ell})$$

- Squared distance from $\hat{\beta}_{k+1,j} = \sqrt{P_\ell}\,\hat{w}_{k,j}$ to $\beta_j = \sqrt{P_\ell}\,1_{\{j=j_\ell\}}$

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Lemma 2

- The power-weighted error and the squared distance have the same expectation.
- Equivalently, the success rate $\sum_{\ell=1}^{L}(P_\ell/P)\hat{w}_{k,j_\ell}$ which is $\beta^T\hat{\beta}_{k+1}/P$ and $\|\hat{\beta}_{k+1}\|^2/P$ have the same expectation.

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Proof: Use $\hat{\beta}_{k+1} = \mathbb{E}\left[\beta | stat_k\right]$.

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Expected success rate: $x_{k+1} = \sum_{\ell=1}^{L}(P_\ell/P)\,\mathbb{E}\left[\hat{w}_{k,j_\ell}\right]$

Consequence for expected success rate

If the expected success rate was x_k, then using the $stat_{k,j}$ representation $\alpha_{\ell,k}\, 1_{\{j=j_\ell\}} + Z_{k,j}$ with

$$\alpha_{\ell,k} = \sqrt{nP_\ell/(\sigma^2 + P(1-x_k))},$$

then at the next step we have

$$x_{k+1} = g(x_k)$$

where $g(x)$ is the success update function

$$g(x) = \sum_{\ell=1}^{L} \frac{P_\ell}{P}\, success(\alpha_\ell(x))$$

where

$$success(\alpha) = \mathbb{E}\left[\frac{e^{\alpha^2+\alpha Z_1}}{e^{\alpha^2+\alpha Z_1} + \sum_{j=2}^{M} e^{\alpha Z_j}}\right]$$

evaluated at $\alpha_\ell(x) = \sqrt{nP_\ell/(\sigma^2 + P(1-x))}$

assuming w.l.o.g. that first term is sent in each section.

Decoding progression

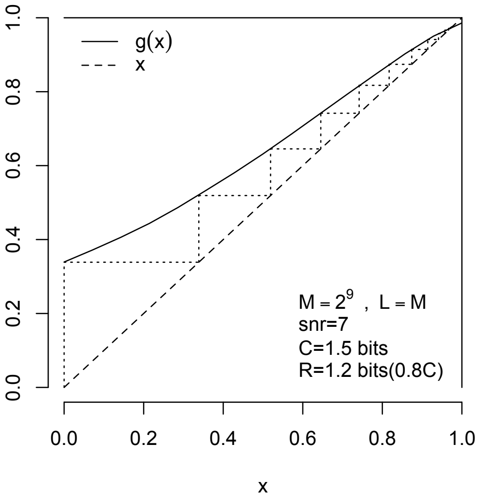


Figure: Plot of $g(x)$ and the sequence x_k.

Integral representation of $g(x)$

Change of variables from $t = \ell/L$ to

$$u = \frac{1 - e^{-2Ct}}{1 - e^{-2C}} \qquad \sim \quad \text{Uniform on } [0, 1],$$

$\alpha_\ell(x)$ becomes

$$\alpha(u, x) \;=\; \tau \sqrt{\frac{C}{R} \; \frac{1 + snr(1-u)}{1 + snr(1-x)}}$$

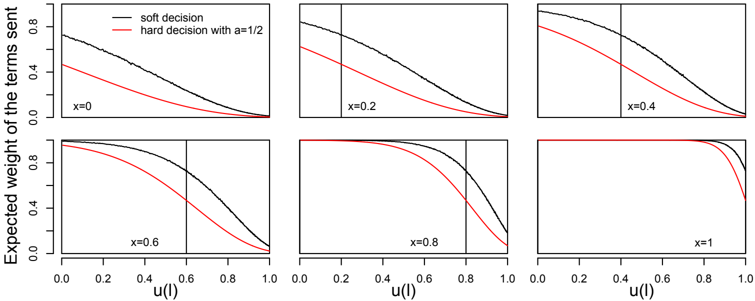
which can be compared to $\tau = \sqrt{2 \log M}$.

We have the integral representation of $g(x)$

$$g(x) = \mathbb{E}_U \left[g(U, x) \right] = \int_0^1 g(u, x) du$$

where $g(u, x) = success(\alpha(u, x))$

Transition plots



Setting: $M = 2^9$, $L = M$, $C = 1.5$ bits and $R = 0.8C$.
Plot of $g(u, x)$ for $x = 0, 0.2, 0.4, 0.6, 0.8, 1$.
Horizontal axis: depicts $u(\ell) = (1 - e^{-2C\ell/L})/(1 - e^{-2C})$.
Black curves: our soft decision decoder
Red curves: thresholding decoder with threshold $\sqrt{2 \log M} + a$

The area under the curve is $g(x)$.

Lowerbound for update function

Using Jensen's inequality, we have

$$
\begin{aligned}
\mathit{success}(\alpha) &= \mathbb{E}\left[\frac{e^{\alpha^2+\alpha Z_1}}{e^{\alpha^2+\alpha Z_1}+\sum_{j=2}^{M} e^{\alpha Z_j}}\right] \\
&\geq \mathbb{E}\left[\frac{e^{\alpha^2+\alpha Z_1}}{e^{\alpha^2+\alpha Z_1}+(M-1)e^{\alpha^2/2}}\right]
\end{aligned}
$$

so that

$$
g(x) \geq \mathbb{P}\{\xi \leq \alpha_U^2/2 - \tau^2/2 + \alpha_U Z\}
$$

where $\quad \xi \sim \mathit{logistic}(0,1) \quad$ and $\alpha_u = \alpha(u,x)$

The Logit representation

- By McFadden(1974),

  Let $s_1, \ldots, s_m$ be a fixed sequence and ϵ_j be independent Gumbel distributed random variable. Then,

$$\mathbb{P}\{s_1 + \epsilon_1 \geq \max_{2 \leq j \leq m} (s_j + \epsilon_j)\} = \frac{e^{s_1}}{\sum_{j=1}^{m} e^{s_j}}$$

  Thus, we can write g(x) as

$$g(x) = \mathbb{P}\left\{\alpha_U^2 + \alpha_U Z_1 + \epsilon_1 \geq \max_{2 \leq j \leq m} (\alpha_U Z_j + \epsilon_j)\right\},$$

Extreme value representation of the update function

- Using the logit representation: Approximation of the update function

$$g(x) = \mathbb{P}\{V_1 \le \alpha_U\},$$

where

$$V_1 = \max_{2 \le j \le m} \left\{ -\frac{Z_1 - Z_j}{2} + \sqrt{\left[\epsilon_j - \epsilon_1 + \frac{(Z_1 - Z_j)^2}{4}\right]_+} \right\}.$$

- For the lowerbound

$$g(x) \ge \mathbb{P}\{V_2 \le \alpha_U\}$$

where

$$V_2 = -Z_1 + \sqrt{(\tau^2 + 2\xi + Z_1^2)_+}.$$

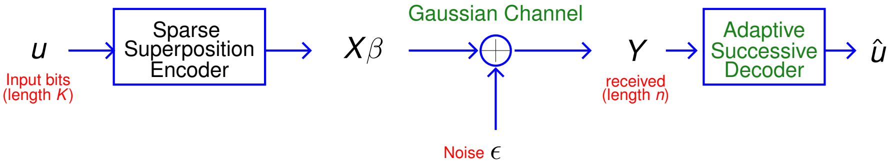
Analysis of Update function

- x^* solves $g(x) = x$, yields mistake rate $1 - x^*$
- Communication rate $R = C/(1 + r/\tau^2)$
- with $r = \mathbb{E}[(V_+^2 - \tau^2)1_B]$, mistake rate

$$1 - x^* = \frac{1}{snr} \, \frac{r}{\tau^2}$$

- Here r grows no faster than order of τ
- $B = \{\alpha(1, x^*) \leq V \leq \alpha(0, x^*)\}$

Summary



Reliable for rates $R < C$

For the adaptive success decoder

- with thresholding (J&B 2010b ISIT, 2012b)
- with iteratively optimal soft decision (shown here)

Update fuctions

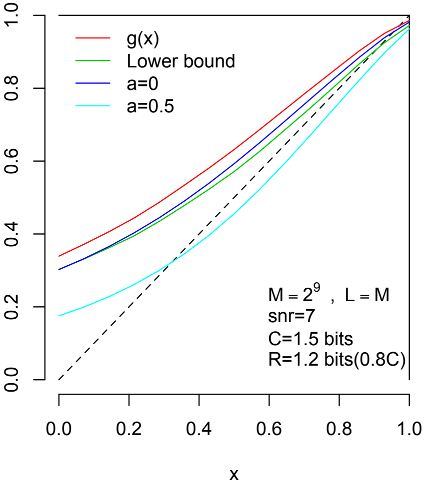


Figure: Comparison of update functions. Blue and light blue lines indicates $\{0, 1\}$ decision using the threshold $\sqrt{2 \log M} + a$ with respect to the value a as indicated.

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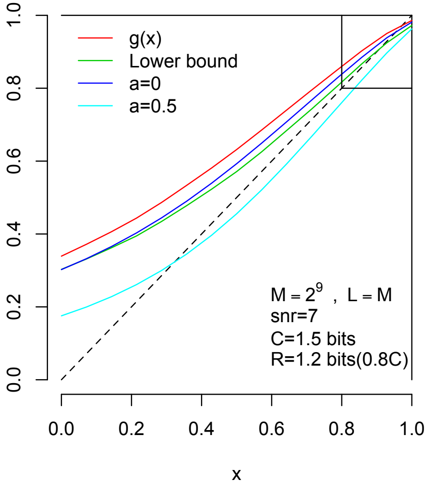

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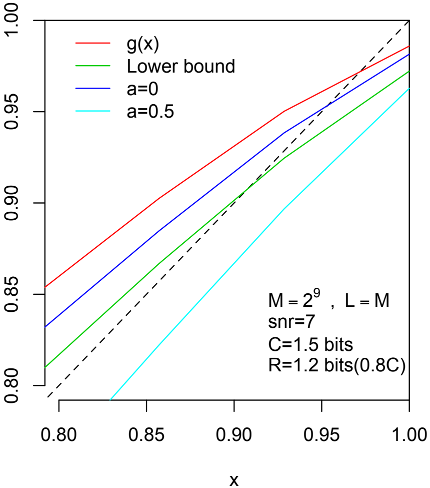

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Iteratively Bayes optimal coefficient estimates

With prior $j_\ell \sim Unif$ on sec_ℓ, the Bayes estimate based on $stat_k$

$$\begin{aligned}\hat{\beta}_{k+1} &= \mathbb{E}[\beta | stat_k] \\ &\cong \mathbb{E}[\beta | stat_k, stat_{k-1}, \ldots, stat_1] \\ &\cong \mathbb{E}[\beta | \mathcal{F}_k]\end{aligned}$$

where $\mathcal{F}_k = \{$ Standardized inner products of X columns with Y and with components of the fits $F_1, \ldots, F_k\}$

Here, $F_k = X\hat{\beta}_k$

Fraction of Mistakes

Translating power-weighted value of $(1 - \hat{w}_{k,j_\ell})$

into fraction of occurrences of $\{1 - \hat{w}_{k,j_\ell} \geq 1/2\}$

$$\delta_{mis} \leq \frac{snr}{C}(1 - x^*) = \frac{r}{C\tau^2}$$

at rate $\quad R = \frac{C}{1+r/\tau^2}$