Maximal Compounded Wealth for Portfolios of Stocks and Options

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Abstract

In the context of multi-period stock market investment with options, we provide characterization of the wealth of constantly rebalanced portfolios of stocks and options. This characterization takes advantage of a correspondence between certain combinations of options and pure gambling opportunities. Through this equivalence, prices to be set for the options correspond to payoff odds on the gambles. Portfolios of a sufficiently complete set of options correspond to betting fraction in gambles on state securities. We use this correspondence to examine the compounded wealth and to show it has a decomposition into a product of three easily interpretable factors. The best portfolio and price strategies with hindsight are identified. We provide universal portfolio strategies that yield the minimax drop in wealth from the maximal compounded wealth for portfolios of stock options.

Key Words: Wealth decomposition, Universal portfolio, Option pricing

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1 Introduction

Consider a portfolio vector $\mathbf{w} = (w_1, w_2, \ldots, w_M)$ with $\sum_{m=1}^{M} w_m = 1$ of M assets, where w_m represents the fraction of an investor's wealth in asset m. For a succession of investment periods $t = 1, 2, \ldots, T$, let $x_{m,t}$ be be the return (wealth factor) for asset m expressed as the ratio of the price at the end of period t to the price at the start of the period, yielding a vector of returns $\mathbf{x}_t = (x_{1,t}, x_{2,t}, \ldots, x_{M,t})$. The wealth in the portfolio in period t is multiplied by $\mathbf{x}_t \cdot \mathbf{w} = \sum_{m=1}^{M} x_{m,t} w_m$. Buying and selling assets each period permits us to rebalance the portfolio so that for the start of the each period w_m is the fraction of the investor's wealth in asset m. Compounding wealth in this way over T periods leads to an T-period wealth factor (ratio of wealth at the end of period T to the wealth at the start of the first period) equal to

$$W_T(\mathbf{w}) = \prod_{t=1}^{T} \mathbf{x}_t \cdot \mathbf{w}. \tag{1.1}$$

Such compounded wealth has been extensively studied from a number of vantage points. Markowitz (1952) initiated the study of portfolio choice via a mean-variance tradeoff. Kelly (1956), Breiman (1961) and Algoet and Cover (1988) explored probabilistic growth rate optimality. Arrow (1964) studied the case of portfolios of state securities in which one can gamble on each possible stock return value. Samuelson (1969) provided a probabilistic utility theory for multi-period investment. Bell and Cover (1980, 1988) examined competitive optimality for arbitrary increasing utilities. Cover (1991), Cover and Ordentlich (1996, 1998), Xie and Barron (2000) and Cross and Barron (2003) established universal portfolios with uniformly small drop in the exponent of wealth from the maximum, uniformly over all possible price sequences.

Before treating the case with options we will review some facts for stock investment in section 2. For any sequence of returns $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_T$, we let $\mathbf{w}^*$ denote a choice that achieves, with hindsight, the maximal compounded wealth $W_T(\mathbf{w}^*) = \max W_T(\mathbf{w})$. We study the drop from the maximum that occurs with fixed $\mathbf{w}$ not equal to $\mathbf{w}^*$ as well as the drop that occurs with certain portfolio estimates.

A special situation, both for its practical interest, and for its mathematical simplicity, is that of the pure gambling scenario, such as a horse race with K horses. In this case the vector of a gambler's betting fractions $\mathbf{b} = (b_0, b_1, \ldots, b_K)$ plays the role of the portfolio vector, where b_k is the fraction of money gambled on horse k and $b_0 = 1 - \sum_{k=1}^K b_k$ is the fraction left in his pocket. Let the odds be c_k for 1 (these odds are also denoted as 1 for $p_k^* = 1/c_k$ or reported as $c_k - 1$ to 1), meaning that if horse k wins then the wealth gambled on that horse is multiplied by c_k. Then after T races, the wealth factor takes the form

$$W_T(\mathbf{b}) = \prod_{t=1}^{T} (b_0 + c_{k_t} b_{k_t})$$

where k_t is the horse that wins race t, for $t = 1, 2, \ldots, T$. Having a positive fraction b_0 reserved for the pocket can be useful when the odds are such that p_k^* sums to more than 1, reflecting a track take. In a sufficiently regulation-free racing market, a no-arbitrage (no free money) argument shows that the odds must satisfy $\sum_{k=1}^K p_k^* = 1$, and whence there is no need for retaining wealth in the pocket as this riskless asset is realizable by a combination of bets on the horses. In that case the compounded wealth $W_T(\mathbf{b})$ takes an especially simple product form

$$W_T(\mathbf{b}) = \prod_{t=1}^{T} c_{k_t} b_{k_t}$$

which may be exactly reexpressed as

$$W_T(\mathbf{b}) = e^{TD(\mathbf{q},\mathbf{p}^*)-TD(\mathbf{q},\mathbf{b})} \tag{1.2}$$

where $D(\mathbf{q}, \mathbf{b}) = \sum_{k=1}^{K} q_k \log(q_k/b_k)$ is the Kullback divergence and where q_k is the relative frequency with which horse k wins the T races. The divergence $D(\mathbf{q}, \mathbf{b})$ is non-negative and it equals zero only when $\mathbf{b}$ equals $\mathbf{q}$. Similar decomposition is possible for the general stock case as we shall see.

What is important in this gambling story is that the wealth identity (1.2) lays bare the roles of choices of the vector of betting fractions $\mathbf{b}$ and of the reciprocal odds $\mathbf{p}^*$ compared to the relative frequency vector $\mathbf{q}$. The wealth is a product of two factors $e^{TD(\mathbf{q},\mathbf{p}^*)}$ and $e^{-TD(\mathbf{q},\mathbf{b})}$. The first governs the impact of the choice of payoff odds and the second reveals the role of the choice of betting fraction $\mathbf{b}$. With hindsight the maximal wealth betting fraction is explicitly $\mathbf{b}^* = \mathbf{q}$, with corresponding maximal wealth $W_T(\mathbf{b}^*) = \max W_T(\mathbf{b}) = e^{TD(\mathbf{q},\mathbf{p}^*)} = W_T^{max}$. Indeed, any $\mathbf{b}$ other than $\mathbf{q}$ yields exponentially smaller wealth by the factor $e^{-TD(\mathbf{q},\mathbf{b})}$. The theory of universal portfolios is simplest in the gambling case and permits solution of estimated portfolios that exactly minimize the worst case drop from the maximal compounded wealth (Cover and Ordentlich (1996), Xie and Barron(2000) building on earlier work by Shtarkov (1988)). Moreover, these minimax strategies achieve a wealth exponent close to the best without prior knowledge of $\mathbf{q}$ uniformly over all possible race outcomes.

The aim of the present paper is to provide similar decomposition of compounded wealth for investment in a stock and in options on that stock.

The use of options with a sufficiently complete set of strike prices enables a dramatic simplification of the stock investment story, both for pricing and for the choice of portfolios and universal portfolio estimates.

In brief, a set of stock options with sufficiently many strike price levels completes the market for that stock to provide opportunity to gamble on the exact state of the stock return. This enables us to provide exact decomposition of the wealth in portfolios of options in terms of the corresponding betting fractions on state securities. A difference from pure gambling is that avoidance of arbitrage restricts the reciprocal odds $\mathbf{p}^*$ to those that make the stock return x be fair, in the sense that $E_{\mathbf{p}^*}x = 1$. Capturing this aspect leads naturally to a wealth decomposition into a product of three factors as revealed in Theorem 4.1. The first factor shows that, even with options, a key role remains for the maximal wealth exponent for the stock and cash alone. The second and third factors express respectively the effects of the choice of price and of portfolio.

Armed with this wealth representation for option investment we provide simple expression for the portfolio of maximum wealth in terms of the relative frequencies of the states of the return. Furthermore, for portfolio estimation these wealth identities with options provide opportunity to determine exact minimax universal portfolios (uniformly over all stock outcome sequences) and to provide explicit easily computed expressions for universal portfolios.

2 Wealth Decomposition

As discussed in the introduction, when gambling on K possible states with relative frequencies q_k, reciprocal odds p_k^* and betting fractions b_k, starting with 1 dollar, the compounded wealth after T gambling periods is

$$W_T^{gambling}(\mathbf{b}) = \prod_{t=1}^{T} b_{k_t} c_{k_t} = e^{TD(\mathbf{q},\mathbf{p}^*)} e^{-TD(\mathbf{q},\mathbf{b})}$$

such that, with hindsight, the best arbitrage-free odds for a bookie are $\mathbf{p}^* = \mathbf{q}$, and, likewise, the best betting fractions are $\mathbf{b} = \mathbf{q}$. Moreover, the difference in the Kullback divergences $D(\mathbf{q}, \mathbf{p}^*)$ and $D(\mathbf{q}, \mathbf{b})$ quantifies the rates of growth of the compounded wealth.

Now let's give analogous conclusions for stock portfolios, followed in the next sections by our main result for portfolios of a stock and options.

Let $\mathcal{X}$ be a set of possible stock return vectors and let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_T$ be the sequence of stock return vectors for T investment periods with each $\mathbf{x}_t$ in $\mathcal{X}$. We consider portfolio vectors $\mathbf{w} = (w_1, w_2, \ldots, w_M)$ with $\sum_{m=1}^{M} w_m = 1$, providing sequences of portfolio returns $\mathbf{w} \cdot \mathbf{x}_1, \mathbf{w} \cdot \mathbf{x}_2, \ldots, \mathbf{w} \cdot \mathbf{x}_T$. We may allow negative w_m (selling short), provided one has the positivity of $\mathbf{x} \cdot \mathbf{w}$ for all possible return vectors $\mathbf{x} \in \mathcal{X}$, which constrains $\mathbf{w}$ to be in a convex set C, here given by $C_{\mathcal{X}} = \{\mathbf{w} : \mathbf{x} \cdot \mathbf{w} \geq 0, \mathbf{x} \in \mathcal{X}, \sum_{m=1}^{M} w_m = 1\}$. If one wishes, one may impose a smaller convex constraint set C, for instance, to prohibit selling short, i.e., $C_+ = \{\mathbf{w} : w_m \geq 0, \sum_{m=1}^{M} w_m = 1\}$.

Each occurrence of a return vector $\mathbf{x}$ in the sequence $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_T$ contributes $1/T$ to the empirical distribution

$$q(\mathbf{x}) = \frac{1}{T} \sum_{t=1}^{T} 1_{\{\mathbf{x}_t = \mathbf{x}\}}.$$

We may write the compounded wealth with portfolio $\mathbf{w}$ as

$$W_T(\mathbf{w}) = \prod_{t=1}^{T} \mathbf{x}_t \cdot \mathbf{w} = e^{Ty(\mathbf{w})}$$

where $y(\mathbf{w}) = \sum_{\mathbf{x}} q(\mathbf{x}) \log(\mathbf{x} \cdot \mathbf{w})$ is a concave function of the portfolio $\mathbf{w}$.

There are degenerate cases with unbounded $C_{\mathcal{X}}$ and $\sup_{\mathbf{w}\in C_{\mathcal{X}}} W_T(\mathbf{w})$ infinite. In appendix A.5, it is shown that a no-arbitrage condition implies that $\sup_{\mathbf{w}\in C_{\mathcal{X}}} W_T(\mathbf{w})$ is finite.

Assume that $\sup_{\mathbf{w}\in C} W_T(\mathbf{w})$ is finite and that there is a maximizer $\mathbf{w}^* = \mathbf{w}^*(\mathbf{q})$ maximizing the compounded wealth over choices of $\mathbf{w}$ in C. If $\mathbf{w}^*$ is in the interior of C, it is characterized by the first derivative condition $\sum_{\mathbf{x}} q(\mathbf{x})x_m/(\mathbf{x}\cdot\mathbf{w}^*) = 1$ for $m = 1, 2, \ldots, M$. More generally (allowing $\mathbf{w}^*$ on the boundary) $\mathbf{w}^*$ maximizes $y(\mathbf{w})$ if and only if $\sum_{\mathbf{x}}(\mathbf{w}\cdot\mathbf{x}/\mathbf{w}^*\cdot\mathbf{x})q(\mathbf{x}) \leq 1$ for all $\mathbf{w}$ in C [Bell and Cover (1980, 1988)].

The Bell and Cover result permits characterization of the wealth in terms of $\mathbf{q}$ and $\mathbf{w}$. A role is played by the non-negative function $q_{\mathbf{w},\mathbf{w}^*}(\mathbf{x}) = (\mathbf{w}\cdot\mathbf{x}/\mathbf{w}^*\cdot\mathbf{x})q(\mathbf{x})$ which we call the wealth drop distribution.

Theorem 2.1: *The compounded wealth of a constant rebalanced portfolio* $\mathbf{w}$ *for* T *periods with relative frequencies of return* $\mathbf{q}$ *is*

$$W_T(\mathbf{w}) = e^{Ty(\mathbf{w})}$$

with exponent

$$y(\mathbf{w}) = D(\mathbf{q}, \mathbf{q}_0) - D(\mathbf{q}, \mathbf{q}_{w,w^*})$$

where $q_{\mathbf{w},\mathbf{w}^*}(\mathbf{x}) = (\mathbf{w}\cdot\mathbf{x}/\mathbf{w}^*\cdot\mathbf{x})q(\mathbf{x})$ *is the wealth drop distribution with* $\sum_{\mathbf{x}} q_{\mathbf{w},\mathbf{w}^*}(\mathbf{x}) \leq 1$ *with equality if* $\mathbf{w}^* \in interior(C)$*, and where*

$$q_0(\mathbf{x}) = q(\mathbf{x})/(\mathbf{w}^*\cdot\mathbf{x})$$

which also sums to not more than 1 when $\mathbf{x}$ *includes a riskless asset of return 1. Thus the wealth has decomposition*

$$W_T(\mathbf{w}) = e^{TD(\mathbf{q},\mathbf{q_0})}e^{-TD(\mathbf{q},\mathbf{q}_{\mathbf{w},\mathbf{w}^*})}$$

The first factor represents the maximal wealth with exponent $y(\mathbf{w}^*) = D(\mathbf{q}, \mathbf{q_0})$ and the second factor represents the drop in wealth from the use of a portfolio $\mathbf{w}$ not equal to $\mathbf{w}^*$. Here, we use $D(\mathbf{q}, \tilde{\mathbf{q}}) = \sum_{\mathbf{x}} q(\mathbf{x}) \log q(\mathbf{x})/\tilde{q}(\mathbf{x})$ defined for non-negative functions $\mathbf{q}$, $\tilde{\mathbf{q}}$. This divergence is non-negative when $\sum_{\mathbf{x}} q(\mathbf{x}) = 1$ and $\sum_{\mathbf{x}} \tilde{q}(\mathbf{x}) \leq 1$ (equalling zero only when $\tilde{\mathbf{q}} = \mathbf{q}$).

Remarks:

1. The maximum growth rate $D(\mathbf{q}, \mathbf{q}_0)$ is non-negative when one of the assets is a "riskless asset" with constant return $c \geq 1$. Then $\sum_{\mathbf{x}} cq_0(\mathbf{x}) \leq 1$ and the maximal growth rate $D(\mathbf{q}, \mathbf{q}_0)$ is at least $\log c$. Larger growth rate requires $E_{\mathbf{q}} x_m$ not equal to the riskless rate c for some variable asset x_m .

2. The wealth $W_T(\mathbf{w}) = \left(\prod_{\mathbf{x}} (\mathbf{x} \cdot \mathbf{w})^{q(\mathbf{x})}\right)^T$ depends on $q(\mathbf{x})$ and $\mathbf{w}$ only through the geometric mean $\prod_{\mathbf{x}} (\mathbf{x} \cdot \mathbf{w})^{q(\mathbf{x})}$ of the portfolio returns. Except in trivial small T cases, it is not a function of the means $E_{\mathbf{q}} x_m$ nor is it a function of the means and the covariances $cov_{\mathbf{q}}(x_j, x_m)$.

3. If $\mathbf{w}$ is chosen to be growth rate optimal for a distribution $\mathbf{p}$, that is, $\mathbf{w} = \mathbf{w}^*(\mathbf{p})$, then the drop from the maximal exponent satisfies $D(\mathbf{q}, \mathbf{q}_{\mathbf{w},\mathbf{w}^*}) \leq D(\mathbf{q}, \mathbf{p})$ [Barron and Cover (1988)].

4. If the coordinates of $\mathbf{w}^*$ and $\mathbf{w}$ are non-negative, then the drop in wealth exponent from use of $\mathbf{w}$ instead of $\mathbf{w}^*$ satisfies

$$D(\mathbf{q}, \mathbf{q}_{\mathbf{w},\mathbf{w}^*}) \leq D(\mathbf{w}^*, \mathbf{w})$$

as proved in Lemma A.6, Appendix A.6.

5. The pure gambling result is a special case in which each return vector $\mathbf{x}$ takes the form $(0, 0, \ldots, c_s, 0, \ldots, 0)$ where the non-zero return is only in one coordinate for one stock s (a "winner-take-all" market). The inequalities from 3 and 4 then show that the worst case drop from the maximal growth rate from using $\mathbf{w}$ other than $\mathbf{w}^*$ occurs in such a horse race case.

6. The maximum wealth portfolio $\mathbf{w}^*$ need not be unique. There can be (especially when the number of stocks is greater than the number of periods) a plane of choices of portfolio vectors $\mathbf{w}$ that yield portfolio returns $\mathbf{w} \cdot \mathbf{x} = \mathbf{w}^* \cdot \mathbf{x}$ for all $\mathbf{x}$ with $q(\mathbf{x}) > 0$. Then $\mathbf{q}_{\mathbf{w},\mathbf{w}^*} = \mathbf{q}$.

7. The portfolio wealth surface $W_T(\mathbf{w})$ can be rather flat function of $\mathbf{w}$ corresponding to $D(\mathbf{q}, \mathbf{q}_{\mathbf{w},\mathbf{w}^*})$ small, e.g., in the case that the portfolio returns are close to each other. This happens in particular if the range of the possible stock returns is small. In contrast, we saw that the gambling wealth surface $W_T^{gambling}(\mathbf{b})$ is very peaked for $\mathbf{b}$ near $\mathbf{q}$ and drops off rapidly away from $\mathbf{q}$. Indeed, the drop in wealth from the maximum is the largest in the gambling case.

8. Potential flatness of $W_T(\mathbf{w})$ surface means that precise estimation of $\mathbf{w}^*$ is not necessarily critical. Historically, it has been common for practitioners to be satisfied with approximation to the wealth surface based on mean-variance tradeoffs, or other utility optimizations. While that may be acceptable for certain stock settings with flat $W_T(\mathbf{w})$, in contrast, for the gambling setting in which the wealth surface is highly peaked, use of portfolios which with hindsight are wealth suboptimal can be a financial disaster.

9. Universal portfolios that update portfolio estimates each time step achieve a growth rate as high as the maximum wealth constant rebalanced portfolio $\mathbf{w}^*$ for arbitrary price sequences, as shall be shown in section 8.

3 The Stock Option Setting

Suppose we consider a single stock in the market and let x denote the stock's return, that is, the ratio of its end-of-period price to its current price. Here x is a variable. We assume that there are K possible states for x, denoted as $a_1, a_2, \ldots, a_K$, given in descending order $a_1 > a_2 > \ldots > a_K > 0$. Let a_{K+1} be a positive number less than a_K. We introduce K options, one for each state, where each share of the k^{th} option is for the right to buy a share of stock at the end of the period at a price of a_{k+1} relative to the current stock price, for $k \in \{1, 2, \ldots, K\}$. When the stock state is x, let z_k be the return for option k. Rationally, investors do not exercise the call option if the price is lower than the strike price. Thus the return is $z_k = (x - a_{k+1})^+/v_k$, where v_k is the ratio of current option price per share to the current stock price. The positive part $(x - a_{k+1})^+$ is used to denote that the option return is zero when $x < a_{k+1}$. The vector of option returns is $\mathbf{z} = (z_1, z_2, \ldots, z_K)$.

Let π_k with $\sum_{k=1}^{K} \pi_k = 1$ denote the fraction of money to invest in option k, then $\pi = (\pi_1, \pi_2, \ldots, \pi_K)$ is a portfolio on the K options. It is possible for π_k to be negative, which means that option k is shorted. Though we shall arrange that the option portfolio return $\pi \cdot \mathbf{z}$ is nonnegative for all possible $\mathbf{z}$ (i.e., for all possible x). We also assume there is a riskless asset with constant return 1. Under a no arbitrage condition, there is no need to explicitly hold wealth in a riskless asset or the underlying stock anymore since they can be replicated by the K options. That is, there exist two portfolios $\pi^{riskless} \in R^K$ and $\pi^{stock} \in R^K$ on options such that for all states of the stock $\pi^{riskless} \cdot \mathbf{z} = 1$ and $\pi^{stock} \cdot \mathbf{z} = x$ (see appendix A.3 for details). Hence, any linear combination of $1, x$, and coordinates of $\mathbf{z}$ with coefficients summing to 1 can be realized by a linear combination on $\mathbf{z}$ alone.

4 Compounded Wealth for Portfolios of Stock Options

Note first that the wealth available in rebalancing between a single stock and cash (with return 1) is

$$W_T^{stock}(w) = \prod_{t=1}^{T} (1 - w + w x_t)$$

The return of the stock each period takes values in the set $\{a_1, a_2, \ldots, a_K\}$. It is $x = a_s$ when state s occurs, where $s \in \{1, 2, \ldots, K\}$. For convenience in relating the option story to the gambling situation, we now denote the relative frequencies of occurrences of state s as $q(s)$ (rather than $q(\mathbf{x})$). From Theorem 2.1, the maximum compounded wealth in the stock and cash case (where the maximum is over all w with possible portfolio returns $(1 - w + w a_k)$ assumed to be non-negative)

$$W_T^{stock,max} = \max_w W_T^{stock}(w) = e^{T y^*}.$$

Here, the maximum wealth portfolio weight w^* is non-zero yielding a positive $y^* = y(w^*)$ when $E_{\mathbf{q}} x \neq 1$ (that is, $\sum_{s=1}^{K} q(s) a_s \neq 1$). The maximum occurs at $w^* = w^*(\mathbf{q})$ satisfying the properties that $q_0(s) = q(s)/(1 - w^* + w^* a_s)$ and $a_s q_0(s)$ both sum to 1. This $W_T^{stock,max}$ has a role in our wealth characterization in the case of stock options.

As we mentioned before, after the introduction of the K options, we only need to choose a portfolio π among these options. Importantly, there is a correspondence between the option price ratios v_s, for $s = 1, 2, \ldots, K$ and the odds (c_s for 1) on state securities, and, moreover, for any portfolio on options, there is a corresponding $\mathbf{b}$ on state securities (betting fraction on "horses") such that the option return matches the gambling return, that is,

$$\pi \cdot \mathbf{z} = b_s c_s \qquad \text{when} \qquad x = a_s.$$

That there should be such a correspondence is intuitively sensible when there is a sufficiently rich collection of strike price levels for call (or put) options. For details of the correspondence in the case of call options, see Appendix A.3. The no-arbitrage condition implies that the reciprocal odds $p^*(s) = 1/c_s$ sum to 1 ($\sum_{s=1}^{K} p^*(s) = 1$) and also that $a_s p^*(s)$ sums to 1 ($E_{\mathbf{p}^*} x = 1$) as discussed in Appendix A.1. Suppose we use portfolio π at periods $1, 2, \ldots, T$ with states s_t and corresponding stock return $x_t = a_{s_t}$, and vector of option returns $\mathbf{z}_t$ with element $z_{k,t} = (x_t - x_{k+1})^+ / v_k$. Then, our wealth is

$$W_T(\pi) = \prod_{t=1}^{T} \pi \cdot \mathbf{z_t}.$$

Here, we also allow negative coordinates of π, provided one has the positivity of $\mathbf{z} \cdot \pi$ for each possible return vector $\mathbf{z}$, i.e. the positivity of b_s, for $s = 1, 2, \ldots, K$.

Theorem 4.1: *Under the no arbitrage condition, the compounded wealth in options is a product of three factors*

$$W_T(\pi) = W_T^{stock,max} e^{TD(\mathbf{q}, \hat{\mathbf{p}}^*)} e^{-TD(\mathbf{q}, \mathbf{b})}$$

where $\hat{p}^*(s) = (1 - w^* + w^* a_s) p^*(s)$, *which gives* $D(\mathbf{q}, \hat{\mathbf{p}}^*) = 0$ *only when the odds* $p^*(s)$ *match* $q(s)/(1 - w^* + w^* a_s)$.

Hence, the maximum wealth in the stock and its options is

$$W_T(\pi^*) = W_T^{stock,max} e^{TD(\mathbf{q}, \hat{\mathbf{p}}^*)}$$

where 1 for $p^*(s)$ are the odds for state securities corresponding to the option prices, for $s = 1, 2, \ldots, K$.

The first factor is the maximum wealth achievable investing in stock and cash only. The second factor is a higher exponential growth available precisely when the option prices are such that $(1 - w^* + w^* a_s)p^*(s)$ is not equal to the relative frequencies $q(s)$, that is, when the state security reciprocal odds $p^*(s)$ are not set to be equal to $q_0(s) = q(s)/(1 - w^* + w^* a_s)$.

The third factor $e^{-TD(\mathbf{q},\mathbf{b})}$ quantifies the drop in wealth by the use of an option portfolio π corresponding to $\mathbf{b}$ on state securities other than the relative frequencies $\mathbf{q}$.

Consequences of Theorem 4.1:

1. Options provide opportunities for greater wealth than with stock and cash alone because of the positivity of the divergence $D(\mathbf{q}, \hat{\mathbf{p}}^*)$ when prices are set with $p^*(s)$ not equal to $q(s)/(1 - w^* + w^* a_s)$.

2. Portfolio choice for an investor is reduced, in the case of options, to the matter of choosing betting fractions $\mathbf{b}$ on state securities to be close to what he believes $\mathbf{q}$ is likely to be.

3. An investor who has confidence in his belief that the relative frequencies will be close to $\mathbf{b}$, is on one hand, encouraged to take the advantage of the options because it produces a higher growth rate by the amount $D(\mathbf{q}, \hat{\mathbf{p}}^*)$. On the other hand, in the case of well-priced options, his drop $D(\mathbf{q}, \mathbf{b})$ from the maximal exponent is greater than the drop $D(\mathbf{q}, \mathbf{q}_{w,w^*})$ in the stock-cash case with $w = w^*(\mathbf{b})$ in accordance with Lemma A.6 (c.f., remarks 3 and 4 of the previous section). Then the investor is better off with the stock-cash rebalancing alone. So if you trust that options are well-priced, you should

not invest in them.

4. In the absence of knowledge of $\mathbf{q}$, one may again seek a sequence of universal portfolios for the options. The reduction of the option problem to a gambling problem provides opportunities to resolve the exact minimax optimal wealth regret (uniformly over all possible outcome sequence) as well as its asymptotics as we discuss in Section 9. These universal portfolios achieve close to the same exponent as an investor who happened to have used a portfolio with fractions $\mathbf{b}$ on the state securities equal to $\mathbf{q}$.

5. Options pricing theory is made general and simple, without requirement of continuous time or of log-normality. If one believes the relative frequencies will be near $\mathbf{p}$, then one uses the options prices corresponding to reciprocal odds of the state securities equal to the "neutral" probabilities $p^*(s) = p(s)/(1 - w^* + w^* a_s)$ with $w^* = w^*(\mathbf{p})$. These are neutral in the sense that any portfolio of securities derived from the state securities will have return z with $E_{\mathbf{p}^*} z = E_{\mathbf{p}} z/(1 - w^* + w^* x) = 1$.

6. Our analysis shows that in a no-arbitrage setting, it is unwise for a broker or a firm to provide a succession of simple single period options. The reason is that fortunate investors whose portfolio corresponds to $\mathbf{b}$ near $\mathbf{q}$ would make an exponential growth of wealth off of the broker, unless the broker happens to have chosen $p^*(s)$ which turns out to match $q(s)/(1 - w^* + w^* a_s)$ associated with the relative frequencies.

   Nonetheless, a broker who knows the probability beliefs of his potential investors and who believes that $\mathbf{q}$ will not be close to any of them is encouraged to offer options or associated gambling opportunities, because the ensuing wealth of the investor $W_T^{stock,max} e^{-TD(\mathbf{q},\mathbf{b})}$ will be less than if they had invested in the stock and cash alone (in view of Lemma A.6), lining the pockets of the broker/bookie.

If regulated in a way that limits competition in offering option, a broker or firm may make money primarily off the transaction fees. The no-arbitrage requirement eliminates opportunity for option fees.

7. In summary, options are to be played only if one has reason to believe that the other parties are less informed. Options are not a financial device that should persist in an informed market.

5 Example

Assume there is a riskless asset with constant return 1 and a stock represented by return vector $(1.3, 1.1, 0.8)$ with current price u. We introduce a put option with strike price $1.1u$, then the return of these three assets can be represented by the following matrix

$$X = \begin{pmatrix} 1 & 1 & 1 \\ 1.3 & 1.1 & 0.8 \\ 0 & 0 & 0.3/v \end{pmatrix}$$

where v is the ratio of current put option price to the current stock price and each row is the return of a security in three different states. Obviously, this return matrix is invertible and it turns out that its inverse is

$$X^{-1} = \begin{pmatrix} -5.5 & 5 & 5v \\ 6.5 & -5 & -25v/3 \\ 0 & 0 & 10v/3 \end{pmatrix}$$

The no-arbitrage condition implies that $\mathbf{x} \cdot \mathbf{p}^* = 1$ for each row of X, thus, $X\mathbf{p}^* = (1, 1, 1)'$. Hence, the state price vector as a function of v is

$$\mathbf{p}^*(v) = X^{-1} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -0.5 + 5v \\ 1.5 - 25v/3 \\ 10v/3 \end{pmatrix}$$

Minimizing $D(\mathbf{p_0}, \mathbf{p}^*)$ will find that the optimal price ratio v is $v^* = 0.1454017$, given the belief $\mathbf{p_0} = (1/3, 1/3, 1/3)$. Also, it is easy to find $w^*(\mathbf{p_0}) = 1.56125$. From the equality, $(1 - w^*(\mathbf{p_0}) + w^*(\mathbf{p_0})a_s)p^*(s) = 1/3$, one may again confirm that the optimal v is $v^* = 0.1454017$. The optimal growth rates for the stock-cash only case and the stock option case are the same with value $y(\mathbf{w}^*) = 0.05163344$ if $\mathbf{p}_0$ is indeed the realized frequency. If $\mathbf{q} \neq \mathbf{p}_0$ is the truly realized frequency, the growth rates are different from 0.05163344. But the rates for the stock-cash only case and the option case are still the same which can be seen from the following section.

6 Gambling Interpretation For Pure Stock Investment

Lemma 6.1: *Let w be the fraction of wealth in the stock in the pure stock-cash investment, let $\mathbf{b}$ be a betting fractions on state securities that realizes a stock-cash portfolio $\mathbf{w}$, and let $\mathbf{q}$ be the relative frequencies of the states. Consider the case, most favorable to the oddsmaker that odds are set such that $p^*(s)(1 - w^* + w^*a_s) = q(s)$. Then the distribution $\mathbf{q}_{\mathbf{w},\mathbf{w}^*}(\mathbf{x}) = \frac{\mathbf{w} \cdot \mathbf{x}}{\mathbf{w}^* \cdot \mathbf{x}} q(\mathbf{x})$ with $\mathbf{w}^* = \mathbf{w}^*(\mathbf{q})$ takes the simple form*

$$\mathbf{q}_{\mathbf{w},\mathbf{w}^*} = \mathbf{b}$$

Proof:

Matching the stock-cash portfolio return $1-w+wa_s$ with the gambling return $b_s c_s = b_s/p^*(s)$, we have

$$b_s = (1 - w + wa_s)p^*(s).$$

The oddsmaker's choice, best with hindsight, is

$$p^*(s) = q(s)/(1 - w^* + w^* a_s).$$

Thus

$$b_s = \frac{1 - w + wa_s}{1 - w^* + w^* a_s} q(s) = \mathbf{q}_{\mathbf{w},\mathbf{w}^*(\mathbf{q})}(s)$$

which completes the proof.

The above lemma give a gambling interpretation for pure stock-cash investment. From this lemma, we can also see that the drops in both cases (pure stock-cask case and the stock option case) from the maximal wealth are the same. What's more, if both the odds maker and the investor make their decision based on a common belief $\mathbf{p}$, then the drops in both cases are the same. Indeed, the drop in the stock-cash case with $w = w^*(\mathbf{p})$ is

$$D(\mathbf{q}, \mathbf{q}_{w^*(\mathbf{p}),w^*(\mathbf{q})}) = \sum_{k=1}^{K} q(k) \log \frac{1 - w^*(\mathbf{q}) + w^*(\mathbf{q})a_k}{1 - w^*(\mathbf{p}) + w^*(\mathbf{p})a_k}$$

Likewise, the drop in the option case using $\mathbf{b}^*(\mathbf{p}) = \mathbf{p}$ and $p^*(s) = p(s)/(1-w^*(\mathbf{p})+w^*(\mathbf{p})a_s)$ is

$$D(\mathbf{q}, \mathbf{b}^*(\mathbf{p})) - D(\mathbf{q}, \hat{\mathbf{p}}^*) = \sum_{k=1}^{K} q(k) \left[\log \frac{q(k)}{b^*(k)} - \log \frac{q(k)}{(1 - w^*(\mathbf{q}) + w^*(\mathbf{q})a_k)p^*(k)} \right].$$

Using $\hat{p}^*(s) = (1 - w^*(\mathbf{q}) + w^*(\mathbf{q})a_s)p^*(s)$, this difference is

$$\sum_{k=1}^{K} q(k) \log \frac{1 - w^*(\mathbf{q}) + w^*(\mathbf{q})a_k}{1 - w^*(\mathbf{p}) + w^*(\mathbf{p})a_k}.$$

which also reduces to the same as for the stock-cash case.

Thus, there is a difference between the stock portfolio and option portfolio cases only when the oddsmaker (option price setter) and the investor hold different beliefs concerning the relative frequencies.

7 Bayes Portfolio Estimates

Let a family $p(\mathbf{x}|\theta)$ of probability densities be given, with θ in some parameter space Θ. For the purpose of construction of some interesting portfolio estimates, assume for now that $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_T$ are conditionally independent given θ. If θ were known, we would be inclined to choose the portfolio vector $\mathbf{w} = \mathbf{w}(\theta)$ to maximize $y(\mathbf{w}, \theta) = E_{\mathbf{x}|\theta} \log \mathbf{x} \cdot \mathbf{w}$, recognizing that averages of $\log \mathbf{x} \cdot \mathbf{w}$ with respect to the relative frequencies $\mathbf{q}$ which determine the compounded wealth would then be close to the averages with respect to $p(\mathbf{x}|\theta)$ with high probability. General probabilistic growth rate optimality is studied in Algoet and Cover (1988) and competitive optimality for arbitrary increasing utilities of wealth ratios is in Bell and Cover (1980, 1988) for any time horizon T. Here, we will not be dwelling on properties of wealth with a presumed knowledge of the distribution. Rather, we explore wealth consequences for certain portfolio estimators. The aim is to have wealth without hindsight knowledge of $\mathbf{q}$ (or of θ) which will perform nearly as well as if we had such knowledge.

If θ is not known, and assigned a prior distribution $h(\theta)$, then we have a posterior distribution $h(\theta|\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{t-1})$ and a predictive distribution $p(\mathbf{x}_t|\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{t-1}) = \int p(\mathbf{x}_t|\theta) h(\theta|\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{t-1}) d\theta$, which is the ratio of $p_t(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_t) = \int h(\theta) \prod_{j=1}^{t} p(\mathbf{x}_j|\theta) d\theta$

to the corresponding value at time $t-1$. Our Bayes portfolio strategy chooses the portfolio vector estimate

$$\mathbf{w}_t(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{t-1}) = arg\max_{\mathbf{w}} E\left[\log(\mathbf{w} \cdot \mathbf{x}_t) | \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{t-1}\right]$$

where the expectation is with respect to the predictive distribution. Now our portfolio is not constant with time but rather updated as a function of the past for each period. With the sequence of portfolios $\{\mathbf{w}_t(\cdot)\}$, our compounded wealth is $W_T = \prod_{t=1}^{T} \mathbf{x}_t \cdot \mathbf{w}_t(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{t-1})$.

Our contribution to the Bayes portfolio story is to note an important class of Bayes strategies in which the role of the parameter is played directly by the portfolio $\mathbf{w}$. Indeed, let any distribution $p_0(\mathbf{x})$ be fixed and let

$$p(\mathbf{x}|\mathbf{w}) = \frac{\mathbf{x} \cdot \mathbf{w}}{\mathbf{x} \cdot \mathbf{w}^*(p_0)} p_0(\mathbf{x})$$

where $\mathbf{w}^*(p_0)$ in C is the p_0-growth optimal portfolio, for which $p(\mathbf{x}|\mathbf{w})$ sums (or integrates) to not more than 1 for all $\mathbf{w}$ in C. With prior $h(\mathbf{w})$ on C, it has a predictive distribution

$$p_t(\mathbf{x}_t|\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{t-1}) = \frac{\int p(\mathbf{x}_t|\mathbf{w}) \left(\prod_{j=1}^{t-1} \mathbf{x}_j \cdot \mathbf{w}\right) h(\mathbf{w}) d\mathbf{w}}{\int \left(\prod_{j=1}^{t-1} \mathbf{x}_j \cdot \mathbf{w}\right) h(\mathbf{w}) d\mathbf{w}} = \frac{\mathbf{x}_t \cdot \mathbf{w}_t(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{t-1})}{\mathbf{x}_t \cdot \mathbf{w}^*(p_0)} p_0(\mathbf{x}_t)$$

where

$$\mathbf{w}_t(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{t-1}) = \frac{\int \mathbf{w} W_{t-1}(\mathbf{w}) h(\mathbf{w}) d\mathbf{w}}{\int W_{t-1}(\mathbf{w}) h(\mathbf{w}) d\mathbf{w}}.$$

Here the choice of $p(\mathbf{x}|\mathbf{w})$ is such that the $p_0(\mathbf{x})/\mathbf{x} \cdot \mathbf{w}^*$ factors have cancelled out of the numerator and denominator, such that $\mathbf{w}_t(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{t-1})$ is the posterior mean of w. Indeed, the role of the likelihood function $\prod_{j=1}^{t-1} p(\mathbf{x}_j|\mathbf{w})$ is played by the wealth function $W_{t-1}(\mathbf{w})$.

Lemma 7.1:

(a) *The portfolio* $\mathbf{w}$ *maximizes the expected log return for the distribution* $p(\mathbf{x}|\mathbf{w})$.

(b) *The Bayes strategy maximizing* $E\left[\log \mathbf{w} \cdot \mathbf{x}_t | \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{t-1}\right]$ *uses the portfolio estimate*

$$\mathbf{w}_t(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{t-1}) = \frac{\int \mathbf{w} W_{t-1}(\mathbf{w}) h(\mathbf{w}) d\mathbf{w}}{\int W_{t-1}(\mathbf{w}) h(\mathbf{w}) d\mathbf{w}}.$$

(c) *The wealth of Bayes strategy* $W_T^h = \prod_{t=1}^T \mathbf{x}_t \cdot \mathbf{w}_t(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{t-1})$ *satisfies*

$$W_T^h = \int h(\mathbf{w}) W_T(\mathbf{w}) d\mathbf{w}.$$

The portfolio in (b) was introduced and studied by Cover (1991) and Cover and Ordentlich (1996) for its universal wealth properties. A new contribution in Lemma 7.1 is the recognition that it arises as Bayes optimal for the families $p(\mathbf{x}|\mathbf{w})$. [Further new contribution to universal portfolio theory will arise from our wealth identities with options in following sections.]

A nice feature of the Bayes portfolio estimates with model $p(\mathbf{x}|\mathbf{w}) = \frac{\mathbf{x} \cdot \mathbf{w} p_0(\mathbf{x})}{\mathbf{x} \cdot \mathbf{w}^*}$ is that the procedure does not depend on $p_0(\mathbf{x})$. This gives the theoretical advantage that one may regard $\{\mathbf{w}_t\}$ as simultaneously Bayes for many families. Indeed, if in fact the outcomes $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_T$ have a particular distribution $p(\mathbf{x})$, then one may regard our Bayes portfolio estimate as arising from the family $p(\mathbf{x}|\mathbf{w})$ with $p_0 = p$ containing the true distribution at $\mathbf{w} = \mathbf{w}^*(p)$ even though the true p is unknown. That is, there is no model misspecification.

The wealth $W_T^h = \int h(\mathbf{w}) W_T(\mathbf{w}) d\mathbf{w}$ in the discrete prior case becomes $W_T^h = \sum_{\mathbf{w}} h(\mathbf{w}) W_T(\mathbf{w})$ which has the interpretation of a unit of wealth assigned according to $h(\mathbf{w})$ to various fund managers each of who contracts to maintain a prospectus with assigned portfolio $\mathbf{w}$. Then we regather (sum) our wealths $h(\mathbf{w}) W_T(\mathbf{w})$ from each fund at the end of period T, to yield $W_T^h = \sum_{\mathbf{w}} h(\mathbf{w}) W_T(\mathbf{w})$.

We now turn our attention to analysis of this wealth

$$W_T^h(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_T) = \int h(\mathbf{w}) W_T(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_T, \mathbf{w}) d\mathbf{w}.$$

8 Universal Portfolios

Here we explore properties of our Bayes optimal wealth that hold universally for all possible return sequences $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_T$. From our wealth identity $W_T(\mathbf{w}) = e^{Ty^*(\mathbf{q})} e^{-TD(\mathbf{q}, \mathbf{q}_{\mathbf{w},\mathbf{w}^*})}$, where $e^{Ty^*(\mathbf{q})} = \max_{\mathbf{w}} W_T(\mathbf{w}) = W_T^*(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_T)$, we have that

$$W_T^h(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_T) = \int h(\mathbf{w}) W_T(\mathbf{w}) d\mathbf{w} = W_T^*(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_T) \int h(\mathbf{w}) e^{-TD(\mathbf{q}, \mathbf{q}_{\mathbf{w},\mathbf{w}^*})} d\mathbf{w}.$$

Our inequality $D(\mathbf{q}, \mathbf{q}_{\mathbf{w},\mathbf{w}^*}) \leq D(\mathbf{w}^*, \mathbf{w})$ then yields

$$W_T^h(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_T) \geq W_T^*(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_T) \int h(\mathbf{w}) e^{-TD(\mathbf{w}^*, \mathbf{w})} d\mathbf{w} \tag{8.1}$$

where the second factor $\int h(\mathbf{w}) e^{-TD(\mathbf{w}^*, \mathbf{w})} d\mathbf{w}$ does not depend on the returns $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_T$. The ratio $W_T^*(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_T)/W_T^h(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_T)$ represents the ratio of the maximal wealth (with hindsight knowledge of $\mathbf{q}$) to the actual wealth achieved by Bayes strategy with prior h. We call this our regret.

From (8.1) the regret is never more than $1/\int h(\mathbf{w}) e^{-TD(\mathbf{w}^*, \mathbf{w})} d\mathbf{w}$. This bound holds with equality when the returns $\mathbf{x}_t$ are winner-take-all (the horse race case). In that extremal case, the bound has been studied in related problem of universal gambling and universal prediction (and universal data compression) in Xie and Barron (2000). There particular mixtures of Dirichlet priors are used (and other choices of priors) to produce a regret that is nearly constant (uniformly over all possible gambling outcomes) and to provide bounds on the gambling regret of the form

$$C_M \cdot \left(\frac{T}{2\pi} \right)^{\frac{M-1}{2}} \tag{8.2}$$

where $C_M = \Gamma(1/2)^M / \Gamma(M/2)$. Thus from inequality (8.1), we have that for all stock return sequences the wealth of a suitable universal Bayes strategy never drops below the maximal wealth $W_T^* = e^{Ty^*(\mathbf{q})}$ (which is exponentially large) by more than this polynomial in T.

Theorem 8.1: *Let h be as in Xie and Barron (2000). Then the Bayes optimal portfolio achieves*

$$\frac{W_T^h(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_T)}{W_T^*(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_T)} \geq \frac{1}{C_M}\left(\frac{2\pi}{T}\right)^{\frac{M-1}{2}} (1 + o(1))$$

uniformly over all stock return sequences $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_T$.

Remarks:

1. Similar bounds, but with a larger constant in place of C_M and without the Bayes interpretation are in Cover and Ordendlich (1996) for the case that h is the $Dirichlet(1/2, 1/2, \ldots, 1/2)$ distribution. Also of interest is the minimax problem

   $$\min_{\{\mathbf{w}_t\}} \max_{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_T} \frac{W_T^*(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_T)}{W_T(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_T, \{\mathbf{w}_t\})} \tag{8.3}$$

   where the minimum is over all sequences of portfolios $\mathbf{w}_t = \mathbf{w}_t(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{t-1})$ mapping past outcomes into portfolio vectors in $C = \{\mathbf{w} : w_m \geq 0, \sum_{m=1}^M = 1\}$. Theorem 8.1 gives the upper bound of equation (8.2) on this minimax value.

2. Exact minimax strategies are also known. Such is available first for the pure gambling (horse race) case from the work of Shtarkov (1988), as shown in Cover and Ordentlich (1996) and Xie and Barron (2000), using the so called normalized maximum likelihood distribution, in which the betting fraction $\mathbf{b}_t(\mathbf{x}_t|\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{t-1})$ are those whose product gives $\mathbf{b}^*(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_T) = \max_{\mathbf{b}} \prod_{t=1}^T \mathbf{b}(\mathbf{x}_t)/C_{T,M}$ where $C_{T,M} = \sum_{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_T} \max_{\mathbf{b}} \prod_{t=1}^T \mathbf{b}(\mathbf{x}_t)$ and the minimax value (with stock return restricted to winner-take-all) is then $C_{T,M}$, for every time horizon T and number of horses M. For large T they show it has asymptotics $C_{T,M} = \frac{1}{C_M}\left(\frac{2\pi}{T}\right)^{\frac{M-1}{2}}(1 + o(1))$. Ordentlich and Cover (1998) determine an exact minimax strategy for the general stock market (with-

out option). The minimax value for all possible stock return vectors is the same as with stocks restricted to winner-take-all.

3. The target level of performance is the best (with hindsight) portfolio $\mathbf{w}^*$ among *constant rebalanced* portfolios $\mathbf{w}$, and one leaves this class to produce Bayes optimal portfolio estimates which involve some dependence on the past to estimate $\mathbf{w}^*$.

4. It is possible to study also compounded wealth in which there is a higher target level of performance. These larger target classes allow certain parameterized strategies that do allow dependence on the past, perhaps through a state variable. Performance bound for universal portfolio in such cases is studies in Cover and Ordentlich (1996) and in Cross and Barron (2003).

5. The Bayes interpretation we have gives here for the universal portfolios endows these portfolio estimates with an equilibrium optimality property of admissibility. No competing portfolio estimator sequence can have everywhere larger average log-wealth uniformly over all θ.

9 Universal Portfolio of a Stock and Options

We recall that the wealth of a constant rebalanced portfolio of a stock and options (at sufficiently many levels) is

$$W_T(\pi) = W_T(x_1, x_2, \ldots, x_T, \pi) = W_T^{stock,max} e^{TD(\mathbf{q},\hat{\mathbf{q}}^*)} e^{-TD(\mathbf{q},\mathbf{b})} = W_T^{option,max} e^{-TD(\mathbf{q},\mathbf{b})}$$

A probability model can put a general multi-Bernoulli distribution $p(x = a_k|\theta) = \theta_k,\ k = 1, 2, \ldots, K$, on the states of the stocks. If the parameter vector θ is known, the growth rate and competitively optimal strategy would be to use the portfolio $\pi = \pi_\theta$ corresponding to $\mathbf{b} = \theta$. If θ is unknown with prior $h(\theta)$, then the Bayes optimal strategy for the period t uses the portfolio corresponding betting fractions $b(x_t|x_1, x_2, \ldots, x_{t-1})$ that match the Bayes predictive distribution $p(x_t|x_1, x_2, \ldots, x_{t-1})$ yielding wealth

$$W_T^h = \int W_T(\pi_\theta) h(\theta) d\theta$$

which equals

$$W_T^h = W_T^{option,max} \int h(\theta) e^{-TD(\mathbf{q},\theta)} d\theta$$

Thus in this stock option setting, our regret $W_T^h / W_T^{option,max}$ is here shown to be precisely the regret $1/\int h(\mathbf{b}) e^{-TD(\mathbf{q},\mathbf{b})} d\mathbf{b}$ of the pure gambling case (which is the previously studied quantity). Again for certain priors, the regret is nearly constant, not depending strongly on the outcome frequencies $\mathbf{q}$. The regret doesn't depend on the odds of the states (nor option prices) and this regret is bounded by a polynomial in T. Moreover, this universal strategy is also average case-optimal for every T.

Now armed with the option-gambling correspondence, we have access via remark 2 of section 8 to the exact minimax regret procedure for investment in a stock and options.

Theorem 9.1: *The exact minimax regret for a stock with options is achieved by using a sequence of portfolios corresponding to joint gambling fractions equal to*

$$b^*(x_1, x_2, \ldots, x_T) = \max_\theta p(x_1, x_2, \ldots, x_T|\theta) / C_{T,K}$$

with minimax value

$$\min_{\pi_1,\pi_2,\ldots,\pi_T} \max_{x_1,x_2,\ldots,x_T} \frac{\max_\pi W_T(x_1, x_2, \ldots, x_T, \pi)}{W_T^h(x_1, x_2, \ldots, x_T)} = C_{T,K}$$

where $C_{T,K} = \sum_{x_1,x_2,\ldots,x_T} \max_\theta p(x_1, x_2, \ldots, x_T|\theta)$. *Its values for large* T *are approximately given by equation* (8.2) *with* K *in place of* M.

Thus we achieve within a polynomial factor the maximal exponential of wealth over all portfolios of stock and options. This holds no matter what sequence of state security odds (that is, no matter what sequence of options prices) are offered, and no matter what sequence of of return outcomes $x_1, x_2, \ldots, x_T$ occur.

10 Closing Remarks

We have made explicit how the wealth of constant rebalanced portfolios of stocks as well as options on a stock depend on the underlying relative frequencies and odds offered for states of the stock.

These identities permit a smooth carryover of precise results for universal gambling and prediction to stock and option cases. This allows a completely nonprobabilistic story, with bounds in which arbitrary stock return outcomes are permitted while retaining exact average case Bayes optimality.

It is fruitful to develop the probabilistic story further. The likelihood interpretation of compounded wealth permits an analysis of statistical efficiency (and negligibility of superefficiency) of Bayes portfolio estimators. Efficiency for cumulative risk across multiple periods as in Barron and Hengartner (1998) is relevant here. Probabilistic analysis reveals in some cases smaller drop in the wealth exponent for Bayes portfolios for typical stock returns than is reflected in the worst case regret. These probabilistic asymptotics will be explored elsewhere.

Appendix

A.1 Odds on state securities must make assets fair

Here, we are going to give an overview of relevant no-arbitrage arguments in the finite-state setting and clarify some facts used in the main text. Some of the arguments in Appendix A.1-A.4 are standard in finance literature [see Huang and Litzenberger (1988) or Ingersoll (1987)].

Suppose we have a finite set $\{1, 2, \ldots, K\}$ of states. M securities are given by an $K \times M$ matrix X, with X_{km} denoting the return of security m in state k, $k = 1, 2, \ldots, K$. Each row of X provides a possible return vector $\mathbf{x}$. We say that there exists an arbitrage opportunity if there are two portfolios $\mathbf{w}$ and $\tilde{\mathbf{w}}$ with returns $\mathbf{w} \cdot \mathbf{x} \geq \tilde{\mathbf{w}} \cdot \mathbf{x}$ for all possible return vectors and $\mathbf{w} \cdot \mathbf{x} > \tilde{\mathbf{w}} \cdot \mathbf{x}$ for some possible return vector. This kind of opportunity enables an investor to make any large amount of money in some state and has return no less than $\tilde{\mathbf{w}} \cdot \mathbf{x}$ in any other states.

Assume that there exists a state security for each state s which has return c_s ($p^*(s) = c_s^{-1}$) if state s occurs, 0 otherwise. For any security A, represented by vector $(a_1, a_2, \ldots, a_K)$, where a_s is the return of security A if state s occurs, under no arbitrage condition, we can claim that

$$\sum_1^K a_s p^*(s) = 1. \tag{10.1}$$

That is, the mean return of the asset using odds assigned to the state securities must be one. Indeed, one first rule out $a_s = 0$, for $s = 1, 2, \ldots, K$, for then it is clear that there would be an arbitrage opportunity. Hence, assume $a_{s_0} \neq 0$, $s_0 \in \{1, 2, \ldots, K\}$. If $\sum_1^K a_s p^*(s) < 1$, construct a portfolio B that contains $a_s p^*(s)$ fraction of state security s, $s = 1, 2, \ldots, K$ and

$1-\sum_1^K a_s p^*(s)$ fraction of asset A. Portfolio B has return $a_s + (1-\sum_1^K a_s p^*(s))a_s \geq a_s$. Hence, portfolio B has greater return than asset A and it is strictly bigger if state s_0 occurs. That is, there is an arbitrage. A similar argument works if $\sum_{s=1}^K a_s p^*(s) > 1$. Hence, equation (10.1) holds. Applying this equation to the stock and the riskless asset in our context, the no-arbitrage condition implies $\sum_{s=1}^K a_s p^*(s) = \sum_1^K p^*(s) = 1$. What's more, from equation (10.1), the arbitrary asset A can be exactly realized by a portfolio $\mathbf{b}$ with $\sum_{k=1}^K b_s = 1$ of state securities. Indeed, we only need to choose $b_s = a_s p_s^*$. The portfolio $\mathbf{b}$ has exactly the same return with asset A.

A.2 Options provide opportunity to construct state securities

Let the option return matrix $\mathbf{Z}$ denoted by a $K \times K$ matrix with elements $z_{ij} = (a_i - a_{j+1})^+/v_j$, that is, each column of $\mathbf{Z}$ is the return of an option in K different states. Since the matrix is upper triangular with the positive diagonal elements, the matrix is nonsingular. That is, it is invertible. For any s, let $\mathbf{e_s}$ denote the row vector in R^K with a 1 in the s^{th} element and 0's elsewhere. Then, using the inverse of $\mathbf{Z}$, there exists a $1 \times K$ vector $\mathbf{w_s}$ and a scale c_s that is the solution to the following equation

$$\mathbf{Z}\mathbf{w}_s' = c_s \mathbf{e}_s' \qquad \text{with} \quad \sum_{k=1}^K w_{s,k} = 1$$

where $\mathbf{w}_s = (w_{s,1}, w_{s,2}, \ldots, w_{s,K})$. Actually, we can solve it and find that $\mathbf{w}_s' = c_s \mathbf{Z}^{-1}\mathbf{e}_s'$ with $c_s = (\mathbf{1}_K \mathbf{Z}^{-1}\mathbf{e}_s')^{-1}$. Thus portfolio $\mathbf{w}_s$ on options yields a state security s which has a return c_s if state s occurs, 0 otherwise. One may think of the state securities with return vector $c_s\mathbf{e_s}$ as horses in a race where horse s yields return c_s if it wins and zero otherwise. As shown above, no arbitrage implies that $\sum_1^K a_s p^*(s) = \sum_1^K p^*(s) = 1$.

A.3 Correspondence between option portfolios and gambling portfolios

Now, we can obtain any portfolio $\pi = (\pi_1, \pi_2, \ldots, \pi_K)$ with $\sum_{s=1}^K \pi_s = 1$ of the call options by letting $(b_1, b_2, \ldots, b_K)' = (\mathbf{w}_1', \mathbf{w}_2', \ldots, \mathbf{w}_K')^{-1}\pi'$ and placing weight b_s on state security portfolio $\mathbf{w}_s$ with $\sum_{s=1}^K b_s = 1$. Obviously,

$$\pi' = W(b_1, b_2, \ldots, b_K)'$$

Hence,

$$\mathbf{Z}\pi' = (c_1 b_1, c_2 b_2, \ldots, c_K b_K)'$$

where $W = (\mathbf{w}_1', \mathbf{w}_2', \ldots, \mathbf{w}_K') = \mathbf{Z}^{-1}\text{diag}(c_1, c_2, \ldots, c_K)$. Hence, we only need to show $\sum_1^K b_s = 1$. Indeed,

$$\mathbf{1}_K \mathbf{b}' = \mathbf{1}_K W^{-1}\pi' = (c_1^{-1}, c_2^{-1}, \ldots, c_K^{-1})\mathbf{Z}\pi' = \mathbf{1}_K \mathbf{Z}^{-1}(\mathbf{e}_1', \mathbf{e}_2', \ldots, \mathbf{e}_K')\mathbf{Z}\pi' = \mathbf{1}_K \pi' = 1$$

Similarly, it can be shown that we can obtain any portfolio $\mathbf{b} = (b_1, b_2, \ldots, b_K)$ with $\sum_{s=1}^K b_s = 1$ of state securities by letting $\pi = (\pi_1, \pi_2, \ldots, \pi_K) = \mathbf{b}W'$ and placing weight π_s on options s with $\sum_{s=1}^K \pi_s = 1$.

Since any security can be realized by state securities through no arbitrage condition, the above argument shows us that it can also be realized by these K options. Specifically, let $\mathbf{b}^{riskless} = (p_1^*, p_2^*, \ldots, p_K^*)$, $\mathbf{b}^{stock} = (a_1 p_1^*, a_2 p_2^*, \ldots, a_K p_K^*)$ and $\pi^{riskless} = \mathbf{b}^{riskless}W$, $\pi^{stock} = \mathbf{b}^{stock}W$, then portfolio π^{stock} and π^{stock} can realize exactly the riskless asset and the stock respectively. Hence, when we make investment, we only need invest among the options.

A.4 Correspondence between option prices and state security odds

The correspondence between current option prices u_s and the odds c_s on state securities s is given as follows. Indeed,

$$\begin{aligned}(c_1^{-1}, c_2^{-1}, \ldots, c_K^{-1}) &= \mathbf{1}_K Z^{-1}(e_1, e_2, \ldots, e_K) \\ &= \mathbf{1}_K Z^{-1} \\ &= (v_1, v_2, \ldots, v_K)\Lambda^{-1}\end{aligned}$$

where the matrix Λ is

$$\Lambda = \begin{pmatrix} a_1 - a_2 & a_1 - a_3 & \ldots & a_1 - a_K & a_1 - a_{K+1} \\ 0 & a_2 - a_3 & \ldots & a_2 - a_K & a_2 - a_{K+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & a_{K-1} - a_K & a_{K-1} - a_{K+1} \\ 0 & 0 & \ldots & 0 & a_K - a_{K+1} \end{pmatrix}$$

which is independent the prices of the options. Consequently,

$$(v_1, v_2, \ldots, v_K) = (c_1^{-1}, c_2^{-1}, \ldots, c_K^{-1})\Lambda$$

Then the prices of these options are

$$(u_1, u_2, \ldots, u_K) = u_0(c_1^{-1}, c_2^{-1}, \ldots, c_K^{-1})\Lambda = u_0(p_1^*, p_2^*, \ldots, p_K^*)\Lambda$$

Therefore, if we determine $p^*(s)$, we can determine the option prices, and vice versa.

A.5 No-arbitrage implies finite maximal wealth

The arbitrage definition here is slightly different with the definition in finance literature. But it would be easy to show that if there is no-arbitrage (by the definition here) implies that there is no arbitrage (by the definition in finance literature, for example, Duffie (2001)) if the return matrix $\mathbf{X}$ is zero.

If there are only finitely many possible state, the no-arbitrage condition implies the finiteness of the maximal wealth.

Proof: The return of the M stock can be represented by a matrix X, with $X_{k,m}$ denoting the return of stock m in state k. Hence, $X_{k,.}$ is the return of the M stocks in state k and $X_{.,m}$ is the returns of stock m in different states k, for $k = 1, 2, \ldots, K$. If $\mathbf{X} = 0$, it is clear that the claim holds. If $\mathbf{X} \neq 0$, then following from Duffie (2001), there exists a strictly positive vector $\psi = (\psi_1, \psi_2, \ldots, \psi_K)$ such that

$$\psi X = \sum_{k=1}^{K} \psi_k X_{k,.} = \mathbf{1}_M$$

where $\mathbf{1}_M$ is a $1 \times M$ vector of 1s. Let $b_k = \psi_k \mathbf{w} \cdot X_{k,.}$, then $\sum_{k=1}^{K} b_k = \mathbf{w} \cdot \sum_{k=1}^{K} \psi_k X_{k,.} = \mathbf{w} \cdot \mathbf{1}_M = 1$ and $b_k \geq 0$ since $\mathbf{w} \cdot X_{k,.} \geq 0$. Hence,

$$y(\mathbf{w}) = \sum_{k=1}^{K} q(k) \log(X_{k,.} \cdot \mathbf{w}) = \sum_{k=1}^{K} q(k) \log(\psi^{-1} b_k) = D(\mathbf{q}, \psi) - D(\mathbf{q}, \mathbf{b}) \leq D(\mathbf{q}, \psi)$$

On the other hand, if all the states s occurred at some investment period t, for $t = 1, 2, \ldots, T$, and there is an arbitrage opportunity, then it can be shown that $\max W_T(\mathbf{w})$ is infinite using similar argument with Appendix A.1.

A.6

Here we prove the claim of Remark 4 in Theorem 2.1. We restrict attention to portfolio vectors which are non-negative and sum to 1. Here, $\mathbf{w}^*$ is the maximum wealth portfolio subject to that constraint.

Lemma A.6: $D(\mathbf{q}, \mathbf{q}_{\mathbf{w},\mathbf{w}^*}) \leq D(\mathbf{w}, \mathbf{w}^*)$

Proof: From the context of Theorem 2.1, we have

$$D(\mathbf{q}, \mathbf{q}_{\mathbf{w},\mathbf{w}^*}) = \sum_{\mathbf{x}} q(\mathbf{x}) \log \frac{\mathbf{w}^* \cdot \mathbf{x}}{\mathbf{w} \cdot \mathbf{x}}$$

Jensen's inequality implies that

$$\log \frac{\mathbf{w} \cdot \mathbf{x}}{\mathbf{w}^* \cdot \mathbf{x}} = \log \left(\sum_{m=1}^{M} \frac{w_m^* x_m}{\mathbf{w}^* \cdot \mathbf{x}} \cdot \frac{w_m}{w_m^*} \right) \geq \sum_{m=1}^{M} \frac{w_m^* x_m}{\mathbf{w}^* \cdot \mathbf{x}} \log \frac{w_m}{w_m^*}$$

Using the fact that when $w_m^* > 0$, $\sum_{\mathbf{x}} q(\mathbf{x}) x_m / (\mathbf{w}^* \cdot \mathbf{x}) = 1$, averaging the above inequality with respect to $q(\mathbf{x})$ yields that

$$\sum_{\mathbf{x}} q(\mathbf{x}) \log \frac{\mathbf{w} \cdot \mathbf{x}}{\mathbf{w}^* \cdot \mathbf{x}} \geq \sum_{m=1}^{M} w_m^* \log \frac{w_m}{w_m^*} \sum_{\mathbf{x}} \frac{q(\mathbf{x}) x_m}{\mathbf{w}^* \cdot \mathbf{x}} = \sum_{m=1}^{M} w_m^* \log \frac{w_m}{w_m^*}$$

Hence, $D(\mathbf{q}, \mathbf{q}_{\mathbf{w},\mathbf{w}^*}) \leq D(\mathbf{w}^*, \mathbf{w})$.

A.7

Proof of Theorem 2.1:

$$W_T(\mathbf{w}) = \prod_{t=1}^{T} \mathbf{x}_t \cdot \mathbf{w} = e^{T \cdot \frac{1}{T} \sum_{t=1}^{T} \log(\mathbf{x}_t \cdot \mathbf{w})} = e^{T \sum_{\mathbf{x}} q(\mathbf{x}) \log(\mathbf{x} \cdot \mathbf{w})} = e^{T y(\mathbf{w})}$$

where $y(\mathbf{w}) = \sum_{\mathbf{x}} q(\mathbf{x}) \log(\mathbf{x} \cdot \mathbf{w})$.

Note that

$$\log(\mathbf{x} \cdot \mathbf{w}) = \log \left(\frac{q(\mathbf{x})}{q(\mathbf{x})/(\mathbf{x} \cdot \mathbf{w}^*)} \cdot \frac{q(\mathbf{x})(\mathbf{x} \cdot \mathbf{w})/(\mathbf{x} \cdot \mathbf{w}^*)}{q(\mathbf{x})} \right)$$

Averaging the above equality with respect to $q(\mathbf{x})$ yields $y(\mathbf{w}) = D(\mathbf{q}, \mathbf{q}_{\mathbf{0}}) - D(\mathbf{q}, \mathbf{q}_{\mathbf{w},\mathbf{w}^*})$, where $\mathbf{w}^* = \mathbf{w}^*(\mathbf{q})$ satisfying $\sum_{\mathbf{x}} q(\mathbf{x}) x_k / (\mathbf{x} \cdot \mathbf{w}^*) = 1$ for $k = 1, 2, \ldots, K$, $q_{\mathbf{w},\mathbf{w}^*}(\mathbf{x}) = \frac{\mathbf{w} \cdot \mathbf{x}}{\mathbf{w}^* \cdot \mathbf{x}} q(\mathbf{x})$ and $q_0(\mathbf{x}) = q(\mathbf{x})/(\mathbf{w}^* \cdot \mathbf{x})$. Since $q_{\mathbf{w},\mathbf{w}^*}(\mathbf{x})$ is nonnegative and sums not more than 1, it follows that $D(\mathbf{q}, \mathbf{q}_{\mathbf{w},\mathbf{w}^*})$ is greater than or equal to 0 with equality if and only if $\mathbf{q} = \mathbf{q}_{\mathbf{w},\mathbf{w}^*}$. Hence, $y(\mathbf{w})$ is maximized at $\mathbf{w} = \mathbf{w}^*$ and the wealth has decomposition

$$W_T(\mathbf{w}) = e^{T D(\mathbf{q}, \mathbf{q}_{\mathbf{0}})} e^{-T D(\mathbf{q}, \mathbf{q}_{\mathbf{w},\mathbf{w}^*})}$$

Proof of Theorem 4.1:

$$
\begin{aligned}
W_T(\pi) &= \prod_{t=1}^{T} \pi \cdot \mathbf{z_t} \\
&= \prod_{t=1}^{T} b_{s_t} c_{s_t} \\
&= e^{T \cdot \frac{1}{T} \sum_{t=1}^{T} \log b_{s_t} c_{s_t}} \\
&= e^{T \sum_{s=1}^{K} q(s) \log b_s c_s} \\
&= e^{T \sum_{s=1}^{K} q(s) \log\left(\frac{q(s)}{q(s)/(1-w^*+w^* a_s)} \cdot \frac{q(s)}{(1-w^*+w^* a_s) p^*(s)} \cdot \frac{b_s}{q(s)} \right)} \\
(*) &= e^{TD(\mathbf{q},\mathbf{q_0})} e^{TD(\mathbf{q},\hat{\mathbf{p}}^*)} e^{-TD(\mathbf{q},\mathbf{b})} \\
&= W_T^{stock,max} e^{TD(\mathbf{q},\hat{\mathbf{p}}^*)} e^{-TD(\mathbf{q},\mathbf{b})}
\end{aligned}
$$

Since $\sum_{s=1}^{K} p_s^* = \sum_{s=1}^{K} p_s^* a_s = 1$ yields $\sum_{s=1}^{K} \hat{p}_s^* = \sum_{s=1}^{K} (1 - w^* + w^* a_s) p_s^* = 1$, then the Kullback divergences in $(*)$ are in their usual sense. Hence, the maximum wealth in options

$$
\begin{aligned}
W_T(\pi^*) &= W_T^{stock,max} e^{TD(\mathbf{q},\hat{\mathbf{p}}^*)} \\
&= e^{T \sum_{s=1}^{K} q(s) \log \frac{q(s)}{q(s)/(1-w^*+w^* a_s)} + T \sum_{s=1}^{K} q(s) \log \frac{q(s)}{(1-w^*+w^* a_s) p^*(s)}} \\
&= e^{T \sum_{s=1}^{K} q(s) \log \frac{q(s)}{p^*}} \\
&= e^{TD(\mathbf{q},\mathbf{p}^*)}
\end{aligned}
$$

where π^* corresponds to $b^* = q$.

We should also notice that the odds maker does be able to set odds $p^*(s)$ for 1 such that $\hat{p}^*(s) = q(s)$ and $\sum_{s=1}^{K} p^*(s) = \sum_{s=1}^{K} p^*(s) a_s = 1$. Hence, the minmax wealth in options equals $W_T^{stock,max}$. What's more, in this special case, we can prove the existence and uniqueness of w^* directly given the no-arbitrage condition with respect to the states which occurred during the T investment periods.

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