YALE: STATISTICS 542
Some Practice Problems

Problems 1-10 below refer to the following situation. Let $X_1, \ldots, X_n$ be a random sample on X, where X has probability density function

$$f(x;\theta) = kx^3 e^{-x^2/\theta}, \qquad x \geq 0,$$

where k is a constant and θ is an unknown parameter.

1. Show that $k = 2/\theta^2$.

2. Let $Y = 2X^2/\theta$. Prove that $Y \sim \chi_4^2$, and use this result to give the mean and variance of X^2. (Remember that the χ_4^2 distribution is the same as a gamma distribution with $r = 2$ and $\lambda = 1/2$.)

3. Write down the likelihood function based on the sample of size n, and identify a sufficient statistic.

4. Show that the MLE of θ is $\hat{\theta} = \frac{1}{2n}\sum_{i=1}^n X_i^2$. Give its mean and variance, and show that $\hat{\theta}$ is a consistent estimator of θ.

5. Determine whether or not the MLE $\hat{\theta}$ is an efficient estimator of θ.

6. Show that the probability that X takes a value greater than two is $P(X > 2) = g(\theta)$, where

$$g(\theta) \equiv (\frac{4}{\theta} + 1)e^{-4/\theta}.$$

7. Give the MLE of $g(\theta) = P(X > 2)$, and prove that it is a consistent estimator of $P(X > 2)$. (Justify your answer.)

8. Find the mean of X. Using this, give the method of moments estimator $\tilde{\theta}$ of θ based on the sample $X_1, \ldots, X_n$, and show that $\tilde{\theta}$ is a consistent estimator of θ. Is $\tilde{\theta}$ a function of a sufficient statistic?

9. Let $T = \sum_{i=1}^n X_i^2$. Prove that

$$\frac{2T}{\theta} \sim \chi_{4n}^2.$$

10. Show that the most powerful test with significance level α of the null hypothesis $H_0: \ \theta = \theta_0$ against the alternative hypothesis $H_1: \ \theta = \theta_1$, where $\theta_0 > \theta_1$, rejects H_0 if $t \leq k$, where t is the observed value of the statistic T given in Problem 9 above. Explain how the constant k can be found using a table of quantiles of the χ^2 distribution. Is this test a *uniformly most powerful test* of H_0 against the composite alternative hypothesis $H_A: \ \theta < \theta_0$? Why, or why not?

11. An instant lottery ticket has three numbers printed on it, and none, one, two, or all three of these may be "lucky" numbers. A sample of 2, 012 tickets gave the following frequency table:

Number of lucky numbers	0	1	2	4
Frequency	1049	775	178	10

Let the random variable X denote the number of lucky numbers on a ticket. Suggest a probability distribution for X (one of the "standard" distributions). Test whether the data above are consistent with your suggested distribution.

12. Let $X_1, \ldots, X_n$ be a random sample on X, where X has an exponential distribution with parameter λ. Suppose you want to estimate the mean $\theta = 1/\lambda$.

(a) Find the MLE $\hat{\theta}$ of θ, and find its MSE.

(b) Find the MSE of the estimator $a\,\hat{\theta}$, where a is a constant. Find the value a_0 of A which minimizes this MSE. Does the estimator $a_0\hat{\theta}$ beat (dominate) the MLE $\hat{\theta}$?

(c) Find the bias in the estimator $a_0\hat{\theta}$ you found in (b). Is this estimator a consistent estimator of θ?

13. Let X_1 and X_2 be a random sample of size 2 on a random variable X having an exponential distribution with mean $1/\lambda$. Let $\theta = P(X \geq 12) = e^{-12\lambda}$. (Think of X as measuring the lifetime in months of a certain type of lightbulb; then θ is the proportion of these bulbs which last as least a year.)

(a) Give an unbiased estimator of θ based on the first observation X_1.

(b) Clearly $T = X_1 + X_2$ is a sufficient statistic. Find the conditional distribution of X_1 given T.

(c) The Rao-Blackwell Theorem says that an unbiased estimator of θ with smaller variance than the estimator $\hat{\theta}$ you found in (a) is $\tilde{\theta} = E(\hat{\theta}\,|\,T)$, Find $\tilde{\theta}$, and verify that it has smaller variance than $\hat{\theta}$.

14. Let $X_1, \ldots, X_n$ be a random sample on X, where X has pdf

$$f_X(x) = \theta\, x^{\theta-1}, \qquad 0 < x < 1, \qquad (\theta > 0).$$

(a) Give a sufficient statistic.

(b) Find the Cramér-Rao lower bound for the variance of an unbiased estimator of θ.

(c) Find the MLE of θ, and show that it is consistent and asymptotically efficient.

(d) What is the asymptotic distribution of the MLE?

(d) Give the form of an approximate $100(1-\alpha)\%$ confidence interval for θ.

15. Let $X_1, \ldots, X_n$ be a random sample on X, where X has pdf

$$f_X(x) = \theta\, x^{\theta-1}, \qquad 0 < x < 1, \qquad (\theta > 0).$$

(This is the same setup as in Problem 14.)

(a) Show that the MP test with significance level α of $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$, where $\theta_1 > \theta_0$, rejects H_0 if $\hat{\theta} \leq k$, where $\hat{\theta}$ is the MLE. Explain how the constant k can be found using tables of the χ^2 distribution. Is this test a UMP test of H_0 against $H_A : \theta > \theta_0$?

(b) Find, and sketch, the power function of the test in (a).

(c) Determine (as completely as possible) the likelihood ratio (LR) test with significance level α of $H_0 : \theta = \theta_0$ against $H_A^* : \theta \neq \theta_0$.