Assignment 4 Solutions – M. Lacey, 02/18/02

Chapter 8, Problem 5 (part (c))
The asymptotic variance of the maximum likelihood estimate $\hat{p}$ is $\frac{1}{nI(p)}$, where $I(p) = E\left[\frac{\partial}{\partial p} log f(X|p)\right]^2 = -E\left[\frac{\partial^2}{\partial p^2} log f(X|p)\right]$. From part(b), we know that the mle $\hat{p} = \frac{1}{\bar{X}}$. Calculations for $I(p)$:

$$\begin{aligned} f(X|p) &= p(1-p)^{x-1} \\ log f(X|p) &= log p + (x-1) log(1-p) \\ \frac{\partial}{\partial p} log f(X|p) &= \frac{1}{p} - \frac{x-1}{1-p} \\ \frac{\partial^2}{\partial p^2} log f(X|p) &= -\frac{1}{p^2} - \frac{x-1}{(1-p)^2} \end{aligned}$$

Since $E(X) = \frac{1}{p}$, it follows that $I(p) = \frac{1}{p^2(1-p)}$, and thus the asymptotic variance of the mle is $\frac{p^2(1-p)}{n}$.

Chapter 8, Problem 8
Using the normal approximation for the Poisson distribution, the approximate sampling distribution of $\hat{\lambda}$ is $N\left(\lambda_o, \frac{\lambda_o}{n}\right)$. From Example A, we're given that $\hat{\lambda} = 24.9$ and $n = 23$, and using the estimate $s_{\hat{\lambda}}$ the resulting approximate distribution is $N(24.9, 1.08)$. For any value δ,

$$P(|\lambda_o - \hat{\lambda}| > \delta) \approx P\left(|Z| > \frac{\delta}{\sqrt{1.08}}\right) = 2\left(1 - P\left(Z \leq \frac{\delta}{1.04}\right)\right).$$

$$\begin{aligned} \delta = 0.5: \quad P\left(|Z| > \tfrac{0.5}{1.04}\right) &= 0.6307 \\ \delta = 1.0: \quad P\left(|Z| > \tfrac{1.0}{1.04}\right) &= 0.3363 \\ \delta = 1.5: \quad P\left(|Z| > \tfrac{1.5}{1.04}\right) &= 0.1492 \\ \delta = 2.0: \quad P\left(|Z| > \tfrac{2.0}{1.04}\right) &= 0.0545 \\ \delta = 2.5: \quad P\left(|Z| > \tfrac{2.5}{1.04}\right) &= 0.0162 \end{aligned}$$

Chapter 8, Problem 11
In Example D of Section 8.4, $f(x|\alpha) = \frac{1+\alpha x}{2}$, $-1 \leq x \leq 1$ and $-1 \leq \alpha \leq 1$. The method of moments estimate $\hat{\alpha} = 3\bar{X}$.

(a) $E(\hat{\alpha}) = E(3\bar{X}) = 3E(\bar{X})$. Since $E(\bar{X}) = E(X) = \frac{\alpha}{3}$ (this can be easily verified by computing the expectation of X), $E(\hat{\alpha}) = 3(\frac{\alpha}{3}) = \alpha$, and thus the estimate is unbiased.

(b)$Var(\hat{\alpha}) = Var(3\bar{X}) = 9Var(\bar{X}) = 9(\frac{\sigma^2}{n})$. We compute $\sigma^2 = E(X^2) - (E(X))^2 = \frac{1}{3} - (\frac{\alpha}{3})^2$, and it follows that $Var(\hat{\alpha}) = \frac{3-\alpha^2}{n}$.

(c) By the Central Limit Theorem, $\hat{\alpha} \sim N\left(\alpha, \frac{3-\alpha^2}{n}\right)$. For $n = 25$ and $\alpha = 0$, $\hat{\alpha} \sim N(0, 3/25)$, and thus $P(|\hat{\alpha}| > 0.5) \approx P\left(|Z| > \frac{0.5}{\sqrt{3}/5}\right) = 2(1 - P(Z \leq 1.44)) = 0.15$.

Chapter 8, Problem 14 (part (c))

For an i.i.d. sample of random variables with density function $f(x|\sigma) = \frac{1}{2\sigma}exp\left(-\frac{|x|}{\sigma}\right)$, we know from part (b) that the mle $\hat{\sigma} = \frac{1}{n}\sum|x_i|$. To find the asymptotic variance of $\hat{\sigma}$, we need to calculate $I(\sigma) = E\left[\frac{\partial}{\partial\sigma}logf(X|\sigma)\right]^2 = -E\left[\frac{\partial^2}{\partial\sigma^2}logf(X|\sigma)\right]$:

$$\begin{aligned}
logf(X|\sigma) &= -log(2\sigma) - \frac{|x|}{\sigma} \\
\frac{\partial}{\partial\sigma}logf(X|\sigma) &= -\frac{1}{\sigma} + \frac{|x|}{\sigma^2} \\
\frac{\partial^2}{\partial\sigma^2}logf(X|\sigma) &= \frac{1}{\sigma^2} - \frac{2|x|}{\sigma^3} \\
-E\left[\frac{\partial^2}{\partial\sigma^2}logf(X|\sigma)\right] &= E\left(\frac{2|x|}{\sigma^3} - \frac{1}{\sigma^2}\right)
\end{aligned}$$

$E(|x|) = \sigma$, so $I(\sigma) = \frac{1}{\sigma^2}$, and the asymptotic variance $\frac{1}{nI(\sigma)} = \frac{\sigma^2}{n}$.

Chapter 8, Problem 25

Electronic components have lifetimes that are exponentially distributed with density function $f(t|\tau) = (1/\tau)exp(-t/\tau), t \geq 0$. Of five new components which are tested, one fails after 100 days.

(a) Let T be the time of the first failure. The distribution of T is exponential with parameter $\frac{n}{\tau} = \frac{5}{\tau}$ for $n = 5$, so $f(t|\tau) = \frac{5}{\tau}exp\left(\frac{-5t}{\tau}\right)$.

(b) To find the mle, compute

$$\frac{\partial}{\partial\tau}logf(t|\tau) = \frac{\partial}{\partial\tau}\left(log(5) - log(\tau) - \frac{5t}{\tau}\right) = -\frac{1}{\tau} + \frac{5t}{\tau^2} \Rightarrow \hat{\tau} = 5T.$$

(c) To calculate the sampling distribution of the mle $\hat{\tau}$, note that $P(\hat{\tau} \leq t) = P(5T \leq t) = P(T \leq t/5) = 1 - exp\left(-\left(\frac{5}{\tau}\right)\left(\frac{t}{5}\right)\right) = 1 - exp(-t/\tau)$, and thus $f(\hat{\tau}) = (1/\tau)exp(-t/\tau) \Rightarrow \hat{\tau} \sim exp(1/\tau)$.

(d) For an exponential random variable X with parameter λ, $Var(X) = 1/\lambda^2$. Since $\hat{\tau} \sim exp(1/\tau)$, $SE(\hat{\tau}) = \sqrt{\tau^2} = \tau$.

Chapter 8, Problem 44

$X_1, \ldots, X_n$ are i.i.d random variables with density function $f(x|\theta) = (\theta+1)x^\theta, 0 \leq x \leq 1$.

(a) To find the MOME, calculate $E(X) = \int_0^1(\theta+1)x^{(\theta+1)}dx = \frac{\theta+1}{\theta+2} = \bar{X} \Rightarrow \tilde{\theta} = \frac{2\bar{X}-1}{1-\bar{X}}$.

(b) The likelihood function $f(x_1, \ldots, x_n|\theta) = (\theta+1)^n(\prod x_i)^\theta$, the log-likelihood $logf(x_1, \ldots, x_n|\theta) = nlog(\theta+1) + \theta log\prod x_i = nlog(\theta+1) + \theta\sum log(x_i)$, and $\frac{\partial}{\partial\theta}logf(x_1, \ldots, x_n|\theta) = \frac{n}{\theta+1} + \sum log(x_i)$. It follows that the mle $\hat{\theta} = -\frac{n}{\sum log(x_i)} - 1$.

(c) The second derivative of the log-likelihood $\frac{\partial^2}{\partial\theta^2}logf(x|\theta) = \frac{\partial}{\partial\theta}\left(\frac{1}{\theta+1} + log(x)\right) = -\frac{1}{(\theta+1)^2}$, and thus $I(\theta) = -E(\frac{\partial^2}{\partial\theta^2}logf(x|\theta)) = \frac{1}{(\theta+1)^2}$. The asymptotic variance of the mle is given by $\frac{1}{nI(\theta)} = \frac{(\theta+1)^2}{n}$.

(d) By the factorization theorem, $\prod x_i$ is sufficient for θ: let $g[T(x_1, \ldots, x_n), \theta] = (\theta+1)^n(\prod x_i)^\theta$, and let $h(x_1, \ldots, x_n) = 1$.