Theory of Statistics.

Homework VI

February 28, 2002. MT

9.1 The coin flips can be assumed independent realizations from a Bernoulli(p) distribution with probability p of getting a "head". Therefore, the number X of heads obtained as a result of 10 coin flips follows a Binomial$(10, p)$ distribution. For testing the null hypothesis $H_0 : p = 1/2$ versus the alternative hypothesis $H_1 : p \neq 1/2$, the rejection region

$$R = \{x : x = 0 \text{ or } x = 10\}$$

is proposed. (a) To evaluate the significance level of such a test:

$$\alpha = \mathbb{P}_{H_0}[X \in R] = \mathbb{P}_{H_0}(X = 0) + \mathbb{P}_{H_0}(X = 10) = (1/2)^{10} + (1/2)^{10} \approx 2/1000 = 0.002$$

[To avoid using a calculator, recall $2^{10} \approx 1000$]. (b) To evaluate the power of this test when the alternative is $H_1 : p = 1/10$, simply compute

$$1 - \beta = \mathbb{P}_{H_1}[X \in R] = \mathbb{P}_{H_1}(X = 0) + \mathbb{P}_{H_1}(X = 10) = (9/10)^{10} + (1/10)^{10} \approx 0.35$$

9.2 (a) $H : X \sim$ Uniform$[0, 1]$ is a simple hypothesis about the parameter in the family of distributions Uniform$[0, \theta]$. It simply corresponds to testing $\theta = 1$. (b) The hypothesis that a die is unbiased is a simple hypothesis about the probability p of rolling a "3", say, with the understanding that each outcome is equally likely (the die is fair). Thus it corresponds to $H : p = 1/6$. (c) $H : X \sim \mathcal{N}(0, \sigma^2), \sigma^2 > 10$ is a composite hypothesis, since it is about the half line

$$\{(\mu, \sigma^2) \in \mathbb{R} \times \mathbb{R}^+ : \mu = 0, \sigma^2 > 10\} = \{0\} \times (10, +\infty)$$

of the parameter space $\mathbb{R} \times \mathbb{R}^+$. (d) $H : X \sim \mathcal{N}(0, \sigma^2)$ is also a composite hypothesis, since it is about the half line

$$\{(\mu, \sigma^2) \in \mathbb{R} \times \mathbb{R}^+ : \mu = 0\} = \{0\} \times (0, +\infty)$$

of the parameter space. Note that the case $\sigma^2 = 0$ corresponds to a degenerate distribution.

9.3 $X \sim$ Binomial$(100, p)$, therefore $\mathbb{E}X = 100p$ while $\text{Var}(X) = 100p(1 - p)$. To test $H_0 : p = 1/2$ versus $H_1 : p \neq 1/2$, the rejection region

$$R = \{x : |x - 50| > 10\} = \{x : x > 60 \text{ or } x < 40\}$$

is proposed. (a) The significance level α of this test is

$$\alpha = \mathbb{P}_{H_0}[X \in R] = \mathbb{P}_{H_0}\left[\frac{|X - 50|}{5} > 2\right] = \mathbb{P}[|Z| > 2] \approx 0.05$$

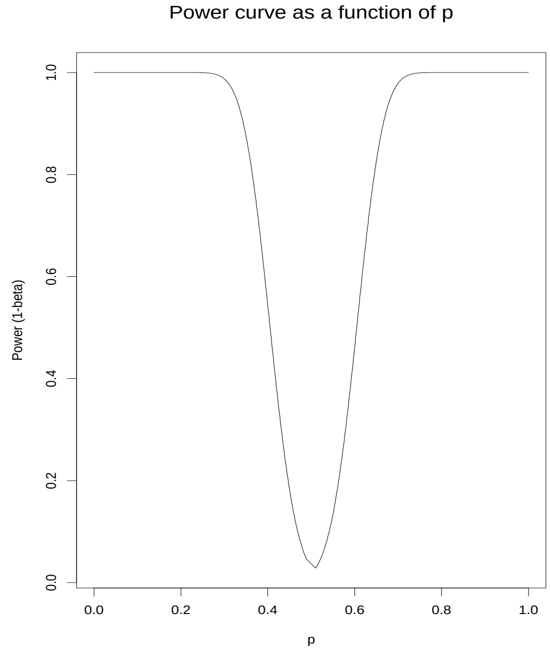

Figure 1: Power curve for the test of $H_0 : p = 1/2$ vs $H_1 : p \neq 1/2$, $R = \{x : |x - 50| > 10\}$.

where $Z \approx \mathcal{N}(0, 1)$. (b) For the power, $1 - \beta = \mathbb{P}_{H_1}[X \in R]$, you can re-express R in terms of $z = \dfrac{(x - 100p)}{10\sqrt{p(1-p)}}$ as follows

$$R = \{z : z > \frac{6}{\sqrt{p(1-p)}} \text{ or } z < \frac{4}{\sqrt{p(1-p)}}\}$$

and then appeal to the fact that $Z \approx \mathcal{N}(0, 1)$. Alternatively, you can use the Binomial quantiles directly. The power curve as a function of p is plotted in figure 1.

9.7 The $(1/n)$-loglikelihood for the Poisson(λ) is given by $\bar{x} \log \lambda - \lambda$, hence

$$-\log(\text{LR}) = \bar{x}(\log \lambda_1 - \log \lambda_0) - (\lambda_1 - \lambda_0),$$

where, for $\lambda_1 > \lambda_2$, both differences $\log \lambda_1 - \log \lambda_0$ and $\lambda_1 - \lambda_0$ are positive. Hence, for large values of $\bar{x}$, $-\log(\text{LR})$ will become positive, which would lead to rejection of $H_0 : \lambda = \lambda_0$ in favor of the alternative hypothesis that $H_1 : \lambda = \lambda_1$ for $\lambda_1 > \lambda_0$. Hence $R = \{x : \bar{x} > C\}$ for some constant C. Since $\sum X_i \sim \text{Poisson}(n\lambda)$, it follows that

$$R = \left\{x : \frac{\sqrt{n}|\bar{x} - \lambda_0|}{\lambda_0} > C\right\}$$

for some other constant C. Note that $\sqrt{n}|\bar{x} - \lambda_0|/\lambda_0$ is approximately $\mathcal{N}(0, 1)$, so C can be chosen so as to make the coverage probability of R be α.

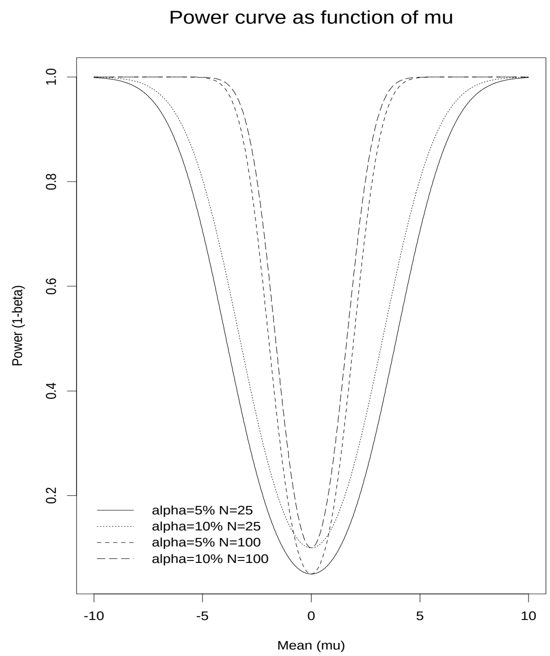

Figure 2: Power curve for the likelihood ratio test of $H_0 : \mu = 0$ vs $H_1 : \mu \neq 0$.

9.8 By Neyman-Pearson, the likelihood ratio test $-\log(\mathrm{LR})$ in 9.7 is most powerful amongst tests which have significance level (less than or equal to) α. Indeed, this is true for every value of $\lambda = \lambda_1 > \lambda_0$. Further, the boundary of R in 9.7 does not depend on λ_1, but only on λ_0 and n. Since R remains the same for every simple alternative $H_1 : \lambda = \lambda_1$ with $\lambda_1 > \lambda_0$, it follows that the test with rejection region R is uniformly most powerful.

9.10 Let $x = (x_1, \cdots, x_n)$. Since T is a sufficient statistic, the likelihood ratio takes the form

$$\frac{f(x \mid \theta_0)}{f(x \mid \theta_1)} = \frac{g(T(x), \theta_0)h(x)}{g(T(x), \theta_1)h(x)} = \frac{g(T(x), \theta_0)}{g(T(x), \theta_1)},$$

which depends on the data only through T. We can therefore construct a rejection region of the form

$$R = \left\{ \frac{g(T(x), \theta_0)}{g(T(x), \theta_1)} < C \right\}.$$

9.11 See figure 2.