Maximal Wealth Portfolios

A Dissertation

Presented to the Faculty of the Graduate School

of

Yale University

in Candidacy for the Degree of

Doctor of Philosophy

by

Wei Qiu

Dissertation Advisor: Andrew R. Barron

May 2007

UMI Number: 3267348

ABSTRACT

Maximal Wealth Portfolios

Wei Qiu

Yale University

May 2007

We analyze wealth maximization for constant rebalanced portfolios. We show how to compute a combination of assets producing an appropriate index of past performance. The desired index is equal to $S_T^{max} = \max_{\underline{b}} S_T(\underline{b})$ which is the maximum of T-period investment return $S_T(\underline{b}) = \prod_{t=1}^{T} \underline{b} \cdot \underline{x}_t$, where $\underline{x}_t$ is the vector of returns for the t^{th} investment period, and $\underline{b}$ is the portfolio vector specifying the fraction of wealth allocated to each asset. We provide an iterative algorithm to approximate this index, where at step k the algorithm produces a portfolio with at most k assets selected among M available assets. We show that the multi-period wealth factor $S_T(\underline{b}_k)$ converges to the maximum S_T^{max} as k increases. Furthermore, the logarithmic wealth factor is within c^2/k of the maximum, where c is determined by the empirical volatility of the stock returns, and we compare this computation to what is achieved by general procedures for convex optimization. This S_T^{max} provides an index of historical asset performance which corresponds to the best constant rebalanced portfolio with hindsight. Surprisingly, we find empirically that a small handful of stocks among hundreds of candidate stocks are sufficient to have come close to S_T^{max}.

Universal portfolios are strategies for updating portfolios each period to achieve

actual wealth with exponent provably close to what is provided by S_T^{max}. We present a new mixture strategy for universal portfolios based on subsets of stocks. Under a volatility condition, this mixture strategy universal portfolio achieves a wealth exponent that drops from the maximum not more than order $\sqrt{\frac{\log M}{T}}$.

ACKNOWLEDGEMENT

I would like to address my most sincere appreciation to Professor Andrew Barron, my thesis advisor, for his enormous amount of help and encouragement he has given to me during the past years. I learned dedication and faith from him and realize that it takes courage to be an excellent researcher.

I would also like to express my ardent gratitude to Professor John A. Hartigan, David Pollard, Joseph T. Chang, Hannes Leeb, Jay Emerson and Mokshay Madiman, from whom I learned a lot by taking classes and having disscussions. Their knowledge and enthusiasm help me to build a solid foundation and lead me into the wonderful world of statistics. I also thank Professor Harry Zhou and Lisha Chen for their careful reading of my dissertation. I thank Joann DelVecchio and Amy L. Mulholland, who helped a lot both in my life and study at Yale.

I want to thank my fellow students. Cong Huang, Rossi Luo and James Hu are among the ones I talked with most. We shared passion and pain during our study at Yale. Jianfeng Yu helped me a lot in the beginning of my graduate study and he also offered great ideas and help in my dissertation work.

At last, but not the least, I want to thank my parents and my wife, Rao Fu. We have gone through many ups and downs in the past years whenever we were together and not. Their emotional support has been necessary for me to complete my study. I also present this dissertation as a gift to our coming baby.

Contents

List of Figures

Chapter 1

Introduction

1.1 Preliminaries

In multi-period investment with a total of M stocks it is important to decide which stocks are to be included in the fund and what fractions of resources are to be allocated to each of them. An investor may choose a portfolio vector $\underline{b}_t$ for trading period t with $\underline{b}_t = (b_{t,1}, b_{t,2}, \ldots, b_{t,M})$ specifying the fraction of wealth to be invested in each of the stocks during that period. Here $b_{t,i}$ represents the proportion of wealth in stock i. One can also put other asset such as cash or bonds in the portfolio. We assume that the portfolio vector satisfies two constraints, $\sum_{i=1}^{M} b_{t,i} = 1$ and $b_{t,i} \geq 0$ for $0 \leq i \leq M$ respectively. The portfolio is self-financing meaning that there is no inflow or outflow of capital required to invest in the portfolio, which is insured by the first constraint. We will not allow investing on margin or short selling of stocks, which means $b_{t,i} < 0$

for some i. This is reflected in the non-negativity constraint.

For a succession of investment periods $t = 1, 2, \cdots, T$ we denote the price per share of stock i at the end of period t by $P_{t,i}$ and the dividend per share during this period to be $Div_{t,i}$. Let $x_{t,i} = (P_{t,i} + Div_{t,i})/P_{t,i-1}$ be the return, also called "wealth factor", for stock i at time t, which means the ratio of the price plus dividend at the end of period t to the price at start of period t. It then provides a vector of returns $\underline{x}_t = (x_{t,1}, x_{t,2}, \cdots, x_{t,M})$ for all M stocks in period t. This $x_{t,i}$ is the multiplicative factor by which wealth in asset i is multiplied. Thus if S_{t-1} is the wealth at the end of period $t-1$ and at the start of period t we buy or sell as needed to have fraction $b_{t,i}$ in stock i. Then the wealth in stock i at the end of the period is $S_{t,i} = S_{t-1} b_{t,i} x_{t,i}$ and the associated total wealth is

$$S_t = S_{t-1}(b_{t,1}x_{t,1} + \ldots + b_{t,M}x_{t,M}).$$

Thus the portfolio $\underline{b}_t$ provides a composite return $\underline{b}_t \cdot \underline{x}_t = \sum_{i=1}^{M} b_{t,i}x_{t,i}$. Then for a sequence of T investment periods with return vector $\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_T$ the compounded multiperiod wealth is

$$S_T = S_0 \prod_{t=1}^{T} \underline{b}_t \cdot \underline{x}_t.$$

With S_0 taken to be on unit of wealth then $S_T = \prod_{t=1}^{T} \underline{b}_t \cdot \underline{x}_t$ is the factor by which our money is multiplied over the T periods.

A constant rebalanced portfolio (CRP) is simply a portfolio that uses the same wealth fractions on stocks each period. It can be interpreted as a buy-low sell-high strategy. That is, when a stock goes down we buy more to retain the desired fraction and likewise sell some of the stock when it goes up. Hence it is different from buy and hold strategies for which no trading is needed during the T investment periods. Any price fluctuation will lead to new asset values at the end of each investment period, in which case the investor trades portions of each asset to restore the specified fractions.

With rebalancing to portfolio $\underline{b}$ each period, the multi-period return or wealth factor (when $S_0 = 1$) is,

$$S_T(\underline{b}) = \prod_{t=1}^{T} \underline{b} \cdot \underline{x}_t.$$

We define $y(\underline{b})$ to be the wealth exponent $\frac{1}{T} \log S_T(\underline{b})$. That is,

$$S_T(\underline{b}) = e^{Ty(\underline{b})}.$$

Closely related to the multi-period return is $(S_T(\underline{b}))^{1/T} = e^{y(\underline{b})}$ which is the empirical geometric mean $(\prod_{t=1}^{T}(\underline{b} \cdot \underline{x}_t))^{1/T}$. This geometric mean provides the equivalent annual compounding rate of return $r(\underline{b}) = e^{y(\underline{b})} - 1$ for which $S_T(\underline{b}) = (1+r(\underline{b}))^T$. We also note that for constant rebalanced portfolio the wealth exponent is the empirical average logarithm return $y(\underline{b}) = \frac{1}{T}\sum_{t=1}^{T} \log \underline{b} \cdot \underline{x}_t$. We call $S_T^{max} = \max_{\underline{b}} S_T(\underline{b})$ the maximal wealth with hindsight and $\underline{b}^{max}$ to be the corresponding maximal wealth portfolio. The associated optimal wealth exponent is $y_{max} = y(\underline{b}^{max})$.

Constant rebalanced portfolios arise in the arbitrary sequence analysis of universal portfolios in Cover (1990), Cover & Ordentlich (1996), and Helmbold et all (1998) and in the stochastic analysis of growth rate optimal portfolios in Kelly (1956), Breiman (1961), Algoet & Cover (1988), of competitively optimal portfolios in Bell & Cover (1980, 1988), and of utility functions in von Neumann & Morgenstern (1944,1947).

We revisit much of this literature further below.

The purpose of this thesis is to provide a provably accurate algorithm for computation of S_T^{max} and $\underline{b}^{max}$ and to provide a mixture strategy for updating portfolios which achieve wealth exponent provably close to y_{max}.

Our analysis uses the arbitrary sequence perspective. We show that the maximal wealth is nearly realized by our mixture strategy for all return sequences with a small drop in the exponent of wealth dependent upon volatility properties of the sequence. In this arbitrary sequence context we are comfortable to regard the analysis as applying to any monotone increasing function (utility) of the multi-period wealth $S_T(\underline{b})$ as all such will share the same target of performance based on S_T^{max} and the associated optimal $\underline{b}^{max}$. In contrast, *expected* utility analysis is quite a different matter. For instance, one can have a dramatically different portfolio maximizing the expectation of a power of $S_T(\underline{b})$ compared to maximizing the expected logarithm of $S_T(\underline{b})$.

There is much previous work in portfolio theory that has focused on the mean-variance criterion and associated efficient frontier, which was formulated by Markowitz (1952) as an optimization problem with quadratic objective and linear constraints. It seeks the portfolio weights that minimize the variance for a given value of mean return or equivalently maximize the mean return for a given variance. In this setting, variance becomes a proxy for risk and the investor tries to maximize expected return for a given level of risk. This forms the basis of the Sharpe-Markowitz theory of investment. Sharp (1985) gives an introduction on this topic. Goetzmann (1996) discusses and gives empirical results for this mean-variance criterion using Standard & Poor 500 stocks, corporate and government bonds and other asset classes over the period 1970 through 1995. In Chapter 5 we compare the wealth achieved by our strategy to that given in Goetzmann. Latane (1967), Hakansson (1971) and Elton & Gruber (1974) discuss maximization of expected geometric mean return. Bernstein and Wilkinson (1997) modified the mean-variance formalism by maximizing geometric mean return with a variance constraint.

Generally, the traditional view of finance has been that an investor shall choose a portfolio by optimizing an expected utility function. Fishburn (1970) and Kreps (1988) provide an introduction. In this literature, a utility function is regarded as resonable if it is both increasing, because more money is better, and concave, because investors are risk adverse. Quadratic utility and exponential utility are among the most commonly used utility functions. Many other types of utility functions have been studied, such as the von Neumann-Morgenstern (1944, 1947) class of utility functions, which includes the power utility $U(s) = (s^\alpha - 1)/\alpha$ for $\alpha < 1$ and the logarithmic utility $U(s) = \log s$. These utilities are distinguished by the property that for $\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_T$ i.i.d. the sequence of portfolio actions best for $EU(S_T)$ is the same as the choice best for $EU(\underline{X}_t \cdot \underline{b})$ each period. The logarithmic utility is shown to produce the highest growth rate of wealth in probability in Algoet and Cover (1988), where if the $\underline{X}_t$ are not i.i.d. the optimal action is to maximize the conditional expected logarithm given the past. In the i.i.d. case one simply observes that the exponent $y_T(\underline{b})$ converge to $E_P \log \underline{X} \cdot \underline{b}$ in probability, so any $\underline{b}$ other than $\underline{b}^* = \arg\max E_P \log \underline{X} \cdot \underline{b}$ will have exponentially smaller growth in probability. Nevertheless, we emphasize that we do not need any stochastic assumptions for the main conclusions of this thesis. We use as the standard of comparison $S_T^{max} = \max_{\underline{b}}(\prod_{t=1}^T \underline{x}_t \cdot \underline{b})$. We show how to compute it for a given sequence of returns and we give strategies for updating investment portfolios which achieve an exponent that matches what S_T^{max} achieves with a drop from the maximum explicitly controlled.

1.2 Summary of the Thesis

We first introduce a wealth maximization algorithm which maximizes the wealth of constant rebalanced portfolio for a given sequence of returns. A maximum wealth asset index is induced by the maximum wealth portfolio. Characteristics of this maximum wealth portfolio reveal historically important stocks and the best fraction of wealth to have retained in each. Moreover, identification of such portfolios from past data may be useful for speculation as to which stocks to invest for subsequent trading periods. We will regard S_T^{max} as an asset index, which refers to the collective performance of a given set of stocks over a given historical time period. Thus we compare S_T^{max} to other indices such as the *Dow Jones Industrial Average*, the *Value Line Index*, and the *Standard & Poor 500 Index*. Our S_T^{max} corresponds to the wealth of the best constant rebalanced portfolio with hindsight.

One of our goals in this thesis is an iterative algorithm for the maximization of $S_T(\underline{b})$, which constructs the portfolio of historically optimal performance. The hindsight maximum wealth at the end of investment period T is $S_T(\underline{b}^{max}) = \prod_{t=1}^{T} \underline{b}^{max} \cdot \underline{x}_t$. We provide an algorithm for this maximization, which chooses stocks from the pool of candidates in a greedy fashion. At the k^{th} step the algorithm either introduces an additional stock to the portfolio or adjusts the weight given to a stock already in the portfolio so as to best balance with the weights of other stocks in the proceeding steps.

Thus the algorithm produces a sequence of portfolios $\underline{b}_k$ where at step k we have included at most k stocks. The multi-period wealth factor $S_T(\underline{b}_k)$, $k = 1, 2, \ldots$ achieved by this sequence of portfolios $\underline{b}_k$ is shown to converge to the maximum $S_T^{\max} = \max_{\underline{b}} S_T(\underline{b})$. In practice we see that it rarely requires more than a few stocks to come close to the maximum. Moreover, we provide theory which shows that an exponent characterizing the wealth at step k is below the maximum by not more than c^2/k for $k = 1, 2, \cdots$. Thus with k stocks we reach approximately the same return as that of the optimal portfolio which has the freedom to have allocated wealth in all the stocks.

Writing $S_T(\underline{b}) = e^{Ty(\underline{b})}$ we find that the wealth exponent $y(\underline{b})$ is concave function of the portfolio $\underline{b}$. Thus we may regard the algorithm provided here as solving a concave optimization problem. We will contrast the method developed here with a general purpose algorithm for maximizing concave functions subject to convex constraint sets (Nesterov and Nemirovski's interior point method (1993)) for which there are also bounds on the number of computation steps required for specified accuracy.

In Chapter 3, we will develop a mixture strategy for universal portfolios that we show achieve a wealth exponent that is within $c\sqrt{(\log M)/T}$ of the maximum where c depends on an empirical relative volatility of the stocks. In the practice of investment one requires a sequence of portfolios $\underline{b}_t$ updated each period t based on what has been observed up to that time. A result of Cover (1991) (refined further in Cover and Ordentlich (1996) and Xie and Barron (2000)) shows that S_T^{max} is achievable by a universal portfolio updating strategy, in the sense that the actual wealth exponent drops from what S_T^{max} achieves by not more than $\frac{M-1}{2T} \log \frac{T}{2\pi} + \frac{c_M}{T}$, uniformly over all possible stock return outcomes, where c_M is a constant. The universal portfolios use at each time t a weighted combination of portfolios $\underline{b}$ weighted by the wealth $S_t(\underline{b})$ up to that time. As we shall discuss, since the drop depends only logarithmically on the number of stock M, our new bound is preferable to the Cover's bound when M is large. Helmbold et al (1998) also show a similar drop when using their portfolio updating rule with learning rate η. However, choosing such a η requires the knowledge of both the number of trading periods T and a lower bound of price return $x_{t,i}$ for all t before starting to invest at time $t = 1$. We devise mixtures that do not require such knowledge in advance.

In Chapter 5, we explore the use of our mixture portfolios imbedded in a strategy for updating our stock portfolios every investment period on actual stock return data.

We also provide a strategy, which uses our wealth maximization algorithm, of selecting past optimal portfolios. In particular, one may use for each month a portfolio equal to the portfolio that made the most wealth with hindsight over a suitable number of preceding months. It shows impressive return compared to returns of other investment strategies, such as the Standard and Poor 500 Index.

Our main results do not need the stochastic assumptions. Nevertheless, in Chapter 6, we will see some interesting results under stochastic assumptions for stock return sequences. We obtain portfolio risk bounds for applying our k-step portfolio realized from historical data to future stock return sequences.

We also discuss the topic of compounded wealth with portfolios of stocks and options in the last chapter. We provide characterization of the wealth of constant rebalanced portfolios of stocks and options. We relate prices of options to payoff odds on the pure gambles and we relate portfolios of a sufficiently complete set of options to betting fractions specified for each possible state of the stocks.

Chapter 2

Wealth Maximization

In this chapter, we will present our wealth maximization algorithm and provide theory for maximum wealth portfolio computation.

2.1 Wealth Maximization Theory

We first introduce a tool for constructing the asset index, namely the computation of $S_T^{max} = \max_{\underline{b}} S_T(\underline{b})$. It is an algorithm, which, when given a series of returns in T periods for M stocks, determines the rebalancing portfolio that would have made the maximal wealth for these stocks in that time frame. We will show that the total computations needed to achieve the targeted accuracy ε by our algorithm is $N_{new}(\varepsilon) = cMT/\varepsilon$. Here c will depend on the sequence of returns $\underline{x}_1, \ldots, \underline{x}_T$ and is not a universal constant. Nevertheless, we argue that for a moderate accuracy ε the typical computation time is such that the cMT/ε is much smaller than the computation time $TM^{4.5}\log(M/\varepsilon)$, guaranteed by an interior point method.

Our algorithm is a multi-step stock selection procedure during which at each step we select one stock from all M stocks. We let S_T^k denote its multiperiod wealth after k steps. The stock selected at step k may be either a stock already selected or a previously unselected stock. The incremental contribution to its portfolio weight is $2/(k+2)$ for $k \geq 2$ (whereas for $k = 1$ we initialize with full weight 1 on the best single stock and 0 on all others), and correspondingly the portfolio weight of an previously selected stock is downweighted by a factor of $k/(k+2)$. This yields a portfolio $\underline{b}_k$ and portfolio returns $Z_{t,k} = \underline{x}_t \cdot \underline{b}_k$ for $t = 1, \ldots, T$ with contribution to the wealth only from the selected stocks. The compounded wealth with this portfolio is $S_T^k = S_T(\underline{b}_k)$. The algorithm is greedy in that at step k the stock $i = i_k$ introduced is the one that (given the $k-1$ previous choices) yields the best such multiperiod portfolio return $\prod_{t=1}^T \left[(1 - \frac{2}{k+2})Z_{t,k-1} + \frac{2}{k+2}x_{t,i}\right]$ that balances the previous portfolio with the newly chosen stock. Here the portfolio return is updated by $Z_{t,k} = \frac{k}{k+2}Z_{t,k-1} + \frac{2}{k+2}x_{t,i_k}$ and its product $S_T^k = \prod_{t=1}^T Z_{t,k}$ is its multiperiod wealth factor. Now we give our wealth maximization theorem.

Theorem 1. *Let a sequence of return vectors $\underline{x}_1, \ldots, \underline{x}_T$ be given and S_T^{max}, S_T^k be defined as above. Our k step algorithm provides a portfolio $\underline{b}_k$ for which*

$$0 \leq \frac{1}{T}\log\frac{S_T^{max}}{S_T^k} \leq \frac{c^2}{k+3} \tag{2.1}$$

or equivalently,

$$S_T^{max} \geq S_T^k \geq S_T^{max} e^{-T\frac{c^2}{k+3}}, \tag{2.2}$$

where $c^2 = 4I\log(2v\sqrt{e})$. *Here* $I = \frac{1}{T}\sum_{t=1}^{T}\sum_{i=1}^{M} b_i^{max}(\frac{x_{t,i}}{\underline{b}^{max}\cdot\underline{x}_t})^2$ *and* $v = \max_{1\leq t\leq T, 1\leq i,j\leq M}$ $\{x_{t,i}/x_{t,j}\}$ *are empirical measures of volatility which depend on the sequence of returns* $\underline{x}_1, \ldots, \underline{x}_T$. *They are constants in the sense that they do not depend on the number of iterations* k.

Concerning the quantities I and v that arise in the definition of c, one may think of I as an average squared empirical relative volatility of only the stocks that arise in the optimal $\underline{b}^{max}$. Likewise $v = \max_{t,i,j} x_{t,i}/x_{t,j}$ is a worst case relative volatility over all candidate stocks. The appearance of this v in the bound is somewhat bothersome but we are pleased that at least it appears only through a logarithm.

To summarize the conclusion of Theorem 1, the wealth that would have been achieved at $\underline{b}_k$ has a drop from the maximal wealth exponent by $\frac{c^2}{k+3}$. Furthermore, S_T^k converges to S_T^{max} as $k \to +\infty$. We emphasize that S_T^{max} and its approximation S_T^k are indices based on historically given return sequences. For future return sequences S_T^{max} can also be conceptualized as a target level of (possibly unachievable) performance.

Further details of the algorithm and the proof of Theorem 1 are given in 2.2.1 and 4.1 respectively.

2.2 Wealth Maximization Analysis

Our historical asset performance index is based on the computational task of wealth maximization, that is, the computation of $S_T^{max} = \max_{\underline{b}} S_T(\underline{b})$ and the determination of the constant rebalancing portfolio $\underline{b}^{max}$ which would achieve this maximum. When given a series of return in T periods for M stocks, this maximization determines the portfolio that would have achieved the maximal wealth for these stocks in that time frame. Hence the maximal wealth factor S_T^{max} shows the index of best past stock performance given the hindsight information. We have related the computation task to concave optimization in 2.3, we will give details and further analysis of the wealth maximization algorithm and the proof of Theorem 1 in 4.1. The advantage of our algorithm is that we only need to consider and optimize a portfolio of two components, which means there is only one parameter to estimate in each step. We first discuss some properties of the target wealth S_T^{max} as the index of historical best performed portfolios with respect to CRP.

2.2.1 Properties of the Target Wealth

As we have already discussed, the CRP is to use portfolio $\underline{b}$ for the first period after which we need to buy and sell appropriate amounts of stocks at the end of each period to insure that the fractions of our wealth are kept fixed as $\underline{b}$ in each stock at the start of the next period. The associated maximal wealth S_T^{max} with portfolio $\underline{b}^{max}$ has some nice properties. The following properties were pointed out by Cover (1990) who viewed S_T^{max} as a target level of performance which universal portfolio aspire to achieve.

First we note that S_T^{max} outperforms the best single stock. Indeed the maximum is taken over all possible constant rebalanced portfolios, which include among them the portfolios concentrated on single stocks. Thus S_T^{max} is larger than the wealth of the best single stock.

It outperforms various stock market indices like the *Dow Jones Industrial Average (DJIA)*. The *DJIA* is a weighted arithmetic average of stock price with corresponding weights $a = (a_1, \ldots, a_M)$ such that $a_i \geq 0$ and $\sum_{i=1}^M a_i = 1$. Let $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 on i^{th} coordinate. So the portfolio $\underline{b} = e_i$ is equivalent to the portfolio with single stock i. Then

$$\begin{aligned} DJIA &= \sum_{i=1}^{M} a_i S_T(e_i) \\ &\leq \sum_{i=1}^{M} a_i S_T^{max} \\ &= S_T^{max} \end{aligned}$$

It also outperforms geometric average indices like the *Value Line Index* as

$$\begin{aligned} ValueLine &= \left(\prod_{i=1}^{M} \frac{S_T(e_i)}{S_0} \right)^{1/M} \\ &\leq \left(\prod_{i=1}^{M} \frac{S_T^{max}}{S_0} \right)^{1/M} \\ &= \frac{S_T^{max}}{S_0} \end{aligned}$$

As we can see S_T^{max} is an index that is greater than *Dow Jones Industrial Average* and the *Value Line Index*. In the following sections, we provide details of our wealth maximization algorithm which is used to approximate S_T^{max}.

2.2.2 Details of the Algorithm

Our algorithm is an iterative procedure to select stocks to put in the portfolio. We select only one additional stock at each step. It is chosen to maximize the wealth of the portfolio that combines the newly selected stock with the portfolio of previously selected stocks. The newly selected one may be among the previously selected stocks (but assigned a new weight) or it may be a new stock not previously selected by the algorithm.

Let S_T^k be the wealth of the newly constructed portfolio at the end of k^{th} step. We know that,

$$S_T^{max} = S_T(\underline{b}^{max}) = e^{Ty^{max}} \tag{2.3}$$

where $y^{max} = y(\underline{b}^{max})$.

We show that for each sequence of stock wealth factor $\underline{x}_1, \underline{x}_2, \cdots, \underline{x}_T$, there is a $c = c(\underline{x}_1, \underline{x}_2, \cdots, \underline{x}_T)$, such that,

$$S_T^k \geq e^{T(y^{max} - c^2/(k+3))} \tag{2.4}$$

By inequality (2.4), we see that after k steps, we are assured a wealth exponent within $c^2/(k+3)$ of the maximum. We will exhibit the form of c. For now we simply emphasize that it does not depend on k.

Portfolios with one stock as occur at the first step of our algorithm correspond to vectors $\underline{b}$ which are non-zero in only one of the M-coordinates, where the weight assigned is trivially $\alpha_1 = 1$. The wealth exponent $y_1(\alpha_1)$ in the single stock case is equal to $\frac{1}{T}\sum_{t=1}^{T}\log x_{t,i}$ where $x_{t,i}$ is the return at time t for the selected stock i. The first step picks the $i = i_1$ among $\{1, \ldots, M\}$ at which the wealth factor S_T^1, is largest. The initial portfolio returns (with $k = 1$) are $Z_{t,1} = x_{t,i_1}$ for $t = 1, 2, \ldots, T$.

Likewise in the second step, given i_1, we select stock i_2 among $\{1, \ldots, M\}$ together with a weight $\alpha_2 \in [0, 1]$ to maximize the resulting wealth which now takes the form

$$\prod_{t=1}^{T}(1 - \alpha_2)x_{t,i_1} + \alpha_2 x_{t,i_2}.$$

The current portfolio $\underline{b}$ is now non-zero in at most two coordinates. The portfolio returns (with $k = 2$) are now $Z_{t,2} = (1 - \alpha_2)x_{t,i_1} + \alpha_2 x_{t,i_2}$ for $t = 1, 2, \ldots, T$. The corresponding stocks in our portfolio after step 2 are i_1 and i_2 (the two could be the same stock).

Generally, in k^{th} step k, we select stock $i = i_k$ among $\{1, \ldots, M\}$ with a weight $\alpha_k \in [0, 1]$ to optimize the wealth

$$\prod_{t=1}^{T}(1 - \alpha_k)Z_{t,k-1} + \alpha_k x_{t,i_k} \tag{2.5}$$

where $Z_{t,k-1}$ is the portfolio return for period t at step $k - 1$. Similarly,

$$Z_{t,k} = [(1 - \alpha_k)Z_{t,k-1} + \alpha_k x_{t,i_k}]. \tag{2.6}$$

After k steps the contribution from a previous step j to the the portfolio weight for stock i_j is

$$\alpha_j \cdot \prod_{m=j+1}^{k}(1 - \alpha_m)$$

for $j = 1, \ldots, k-1$. Therefore, our k-step portfolio $\underline{\alpha}_k$ for the k selected stocks is given as follows:

$$\underline{\alpha}_k = \left(\alpha_1 \cdot \prod_{m=2}^{k} (1-\alpha_m), \ldots, \alpha_{k-1}(1-\alpha_k), \alpha_k \right) \tag{2.7}$$

Recognizing that our procedure permits a stock to be revisited during the selection procedure, we see that the total weight for stock i in the resulting portfolios $\underline{b}_k$ is

$$b_{i,k} = \sum_{j=1}^{k} 1_{\{i_j=i\}} \cdot \alpha_j \prod_{m=j+1}^{k} (1-\alpha_m), \tag{2.8}$$

where at $j = k$ the empty product is interpreted as equal to 1. The associated return is $Z_{t,k} = \underline{b}_k \cdot \underline{x}_t = \sum_{i=1}^{k} b_{i,k} x_{t,i}$ and $y_k(\alpha_k) = y(\underline{b}_k)$. Consequently, our multi-period portfolio return achieved after step k is given by $S_T^k = e^{Ty(\underline{b}_k)}$, where

$$y(\underline{b}_k) = \frac{1}{T} \sum_{t=1}^{T} \log \underline{b}_k \cdot \underline{x}_t = \frac{1}{T} \sum_{t=1}^{T} \log Z_{t,k}. \tag{2.9}$$

We can see that during each step of the iterative procedure, we only consider two components, one of which is the combination of stocks which has already been selected (with their previously determined relative weights), the other one is the newly selected stock from $\{1, \ldots, M\}$. The newly selected stock may be either appearing for the first time or one which has been already selected. In the latter case, the optimization steps serves to adjust the weight of the selected stocks relative to the others. The wealth factor S_T^k after k steps is close to the maximum S_T^{max} in the sense that it has an exponential drop of order $c^2/(k+3)$ as shown in (2.4). Consequently with k increasing, our S_T^k converges to the maximum S_T^{max}.

We will continue the analysis of the wealth maximization algorithm in Section 4.1. Detailed proof of Theorem 1 will be also be given there. In chapter 5, we will use Standard and Poor 500 stocks as our candidates stocks to compute indices for the optimal historical stock portfolio performances for twenty years and for ten years. With these empirical results, we will see that for the optimal portfolio there is only small number of stocks needed in order to achieve the optimal wealth S_T^{max}, even though the optimization has the potential to include all M stocks actively. Indeed, most of the coordinates in $\underline{b}^{max}$ are zero.

In the next section, we will relate our maximization algorithm to general concave optimization. Particularly, we briefly discuss the interior-point method and the barrier method. We will also compare the total number of operations, or computation time, for our algorithm and for the interior-point method.

2.3 Concave Optimization and Computation Time

For any given sequence of returns $\underline{x}_t$, our interest is to determine the portfolio which maximizes $S_T(\underline{b})$, which is equivalent to maximizing the log-wealth function, given by $y(\underline{b}) = \frac{1}{T}\log S_T(\underline{b}) = \frac{1}{T}\sum_{t=1}^{T}\log(\underline{b}\cdot\underline{x}_t)$. We optimize $S_T(\underline{b})$, taking advantage of the fact that $y(\underline{b})$ is a concave function of $\underline{b}$ constrained to the $(M-1)$-dimensioned simplex of values where $b_i \geq 0$ for $i = 1, \ldots, M-1$ and $1 - \sum_{i=1}^{M-1} b_i \geq 0$.

One approach to concave optimization is by existing general purpose algorithms. Consider optimization problems of the following form: $\hat{b} = \arg\max\ y(b)$ where the

n-dimensional parameter b is constrained to a convex set. In particular we may have an optimization problem of the form:

$$\begin{aligned} &\text{maximize} \quad y(b) \\ &\text{subject to} \quad y_i(b) \geq 0, i = 1, \cdots, M, \end{aligned} \tag{2.10}$$

where b is a n dimensional vector and the functions $y, y_1, \cdots, y_M : R^n \to R$ are concave.

2.3.1 Interior Point Methods

Interior-point methods achieve optimization by going through the middle of the solid defined by the problem rather than around its surface. General polynomial algorithms for concave maximization subject to convex constraints have existed since 1976 by Nemirovski and Yudin (1976),(1983) (who developed the ellipsoidal method). Subsequently, Karmarkar (1984) announced a fast polynomial-time interior method for linear programming which is related to classical barrier methods. Later, Nesterov and Nemirovski (1993) extended interior-point theory to cover general nonlinear convex optimization problems. The method of solution involves Newton algorithm steps applied to the objective function with a logarithmic barrier penalty with a particular schedule of values of Lagrange multipliers. For solving the problem (2.10) with a specified accuracy ε, they have shown in (1993) that an ε-solution can be found in no more than

$$N_1(\varepsilon) \leq CM^{1/2} \ln \frac{2MB}{\varepsilon}$$

steps of the preliminary and the main stages. The arithmetic cost of a step does not exceed $O(1)(Mn^2+n^3)$ (there are $O(Mn^2)$ operations to form the Newton system and $O(n^3)$ operations to solve it), so that the total number of operations $N(\varepsilon)$ satisfies

$$N(\varepsilon) \leq CM^{1/2}(Mn^2 + n^3)\ln(\frac{2MB}{\varepsilon}).$$

where C and B are some constants. In their analysis, each call to a subroutine to evaluate the function $y(b)$ or $y_i(b)$ are regarded as one operation. In our stock setting evaluations of $y(b)$ requires T times M elementary operations, where T is the number of time periods and M is the number of stocks. In this case the dimension n and the number of constraints are both of order M, the number of stocks. So the total computation time bound for the interior point method is of order

$$N(\varepsilon) = CTM^{4.5}\log(M/\varepsilon). \tag{2.11}$$

The next section gives addition details concerning the interior-point algorithm of Nesterov and Nemirovski, focusing on the specification of the logarithmic barrier and the computation time. The reader may skip this next section if he wishes as we will not use it in our analysis.

2.3.2 Barrier Method

A barrier method is a particular variation of the interior-point method. The idea of using a barrier and designing barrier methods was studied in the early 1960s by Fiacco and McCormick (1968). These ideas were mainly developed for general non-linear programming. Nesterov and Nemirovski (1993) came up with a special class of such barriers that can be used to encode any convex set. They guarantee that the number of iterations of the algorithm is bounded by a polynomial in the dimension and accuracy of the solution. Karmarkar's (1984) breakthrough revitalized the study of interior point methods and barrier problems, showing that it was possible to create an algorithm for linear programming characterized by polynomial complexity and, moreover, that was competitive with the simplex method.

The first step to solve problem (2.10) using the barrier method is to rewrite the maximization problem by making the constraints implicit in the objective function,

$$\text{maximize} \quad y(\theta) + \sum_{i=1}^{M} (1/t) \log(y_i(\theta) - c_i) \tag{2.12}$$

The objective function above is concave, since $(1/t)\log(u)$ is concave and increasing in u, and twice differentiable. It is the same to consider the equivalent problem by multiplying t to the objective function,

$$\text{maximize} \quad \frac{y(\theta)}{\lambda} - \phi(\theta) \tag{2.13}$$

where $\lambda = 1/t$, θ is constrained to a convex set and $\phi(\theta) = -\sum_{i=1}^{M} \log(y_i(\theta) - c_i)$, which is called the logarithmic barrier or log barrier for the problem (2.10).

For each t in $\mathbf{N}^+$ we define $\theta^*(t)$ as the solution of (2.12) and the set of central points $\{\theta^*(t), t > 0\}$ as central path. Therefore,

$$\begin{aligned} 0 &= t \nabla y(\theta^*(t)) - \nabla \phi(\theta^*(t)) \\ &= t \nabla y(\theta^*(t)) + \sum_{i=1}^{M} \frac{1}{y_i(\theta^*(t)) - c_i} \nabla y_i(\theta^*(t)) \end{aligned} \tag{2.14}$$

Stephen Boyd and Lieven Vandenberghe (2004) give a bound showing how close $y(\theta^*(t))$ is to the optimal value $y(\theta^{max})$. Define,

$$\lambda_i^*(t) = \frac{1}{t(y_i(\theta^*(t)) - c_i)}, i = 1, \cdots, m.$$

Thus (2.14) is equivalent to

$$\nabla y(\theta^*(t)) - \sum_{i=1}^{M} \lambda_i^*(t) \nabla y_i(\theta^*(t)) = 0.$$

We can see that $\theta^*(t)$ maximizes the Lagrangian

$$L(\theta, \lambda) = y(\theta) + \sum_{i=1}^{M} \lambda_i(y_i(\theta) - c_i), \tag{2.15}$$

for $\lambda = \lambda^*(t)$. Thus the following relationship can be seen

$$\begin{aligned} L(\theta^*(t), \lambda^*(t)) &= y(\theta^*(t)) + \sum_{i=1}^{M} \lambda_i^*(t)(y_i(\theta^*(t)) - c_i) \\ &= y(\theta^*(t)) + M/t \\ &\geq y(\theta^{max}) \\ &\geq y(\theta^*(t)). \end{aligned} \tag{2.16}$$

Thus $\theta^*(t)$ converges to the optimal value θ^{max} as $t \to \infty$.

The barrier method is based on solving a sequence of optimization problems, using the last point found as the starting point for the next optimization problem. In other words, we compute $\theta^*(t)$ for a sequence of increasing values of t, until $t \geq M/\varepsilon$, where the ε is the specified accuracy.

A simple algorithm of the method is given as following. Given strictly feasible x, $t := t^{(0)} > 0$, $\mu > 1$, tolerance $\varepsilon > 0$.

Repeat.

1. Outer iteration. Compute $x^i(t)$ by optimizing problem (2.10) with $t = t^{(0)}$ starting at x

2. Update $x := x^i(t)$.

3. Stop when $M/t < \varepsilon$.

4. Increase $t := \mu t$.

As $t_k = \mu^k t_0$ at k^{th} outer iteration, the desired accuracy ε can be achieved after exactly $\lceil \log(M/t^{(0)}\varepsilon)/\log\mu \rceil$ outer iterations. And the complexity theory of Newton's method tells us that the number of Newton steps required for each inner iteration is at most $c + \frac{M(\mu-1-\log\mu)}{\tau}$, where $c = \log(1/\varepsilon)$ and τ is an increasing amount of $y(\theta)$ at each iteration, which depends on backtracking line search constants α and β where $0 < \alpha < 0.5$ and $0 < \beta < 1$. Assume there exists an $c_y > 0$ such that $\nabla^2 y(\theta) \leq c_y I_{M\times M}$ for all $\theta \in \Theta$. For each Newton step the cost of line searches is $l = \max\{\lceil 1 + (\log c_f/\log\beta)\rceil, 1\}$. Therefore, the total number of operations under specified accuracy ε is $N(\varepsilon)$, where

$$N(\varepsilon) = \lceil \frac{\log(M/t^{(0)}\varepsilon)}{\log\mu} \rceil \left(\frac{Ml(\mu - 1 - \log\mu)}{\tau} + cl\right). \tag{2.17}$$

2.3.3 Total Number of Operations

We complete this chapter on discussion of our wealth maximization algorithm by noting its total number of operations, namely, the computation time.

For problem (2.10), our aim is to find a portfolio vector $\underline{b}$ that achieves a value for the wealth factor exponent function $y(\underline{b}) = \frac{1}{T}\sum_{t=1}^{T} \log \underline{b} \cdot \underline{x}_t$ that is within ε of the maximum. Our wealth maximization algorithm after k steps achieves

$$|y(\underline{b_k}) - y(\underline{b}^{max})| \leq \frac{c^2}{k+3},$$

that is $k \geq c^2/\varepsilon - 3$ suffices for the stated aim. During each step, the number of computations equals the number of periods T times the number of candidate stocks M as we search among M stocks for the best one that when combined with the previous step component of stocks has the highest wealth factor for T periods. Therefore, the total number of computations needed to reach the accuracy of ε is $N_{new}(\varepsilon)$, where

$$N_{new}(\varepsilon) \leq TM \cdot \lceil \frac{c^2}{\varepsilon} - 3 \rceil.$$

When compared with (2.11), our procedure is better when the number of stock M is large. However, the number of total computations using barrier method depends on the choices of starting parameters, μ and $t^{(0)}$ and our algorithm performance depends on the shape of stock return $\underline{x}$, that is, the constant c which we know from Theorem 1 is a measure of empirical volatility of stock return $\underline{x}$.

We will not give any more details on the general concave optimization in the thesis, as we focus on our particular result for wealth maximization and mixture strategies. For more details of general concave optimization, Boyd & Vandenberghe (2004) and Nestrov & Nemeirovski (1993) are the definitive sources.

In Section 4.1, we will prove Theorem 1 and will continue to give an extensive detail of our algorithm. In the next chapter, we discuss Cover's universal portfolio which is updated each time period and we relate its exponential growth to that of S_T^{max}. Then we develop our mixture strategies for universal portfolios on subsets on stocks. This mixture strategy replies on the wealth maximization theory and takes advantage of the phenomenon that only a small handful of stocks is needed to reach close to the maximum wealth S_T^{max}.

Chapter 3

Mixture Strategies for Universal Portfolios

Cover's universal portfolio update strategy (1991) gives (initially equal) weights to all portfolios that use all M stocks and each time period updates the weights given to portfolios by the wealth achieved thus far. For Cover's universal portfolio strategy to achieve a nearly maximal wealth exponent, the number of periods T needs to be large compared to M. Moreover, computation of the full mixture is a challenge. Helmbold, et al. (1998) suggested another portfolio sequences which has similar wealth exponent drop bound as our mixture strategy does, however their learning sequences (portfolio sequences) require advance knowledge of the return volatility v. They also suggested a refinement of their learning sequences which then become universal with respect to the constant rebalanced portfolios. However it has a big wealth exponent drop bound.

In this chapter we present new universal portfolios based on mixture of investment on subsets of stocks.

3.1 Universal Portfolios

In order to select the portfolio sequence $\underline{b}^*$ that maximizes the expected logarithm return $E_P \log S_T(\underline{b})$, the process governing stock return $\underline{x}_t$ would need to be known. This seems to be impossible in real investment, as we usually do not know well the distribution of stock returns. Nevertheless, with universal portfolios we attain close to an even higher target $\max_{\underline{b}} \log S_T(\underline{b})$ uniformly over all arbitrary return sequences. The concept of universal portfolios was introduced by Cover (1991), and will be discussed further. We emphasize that a sequence of portfolios updated each period based only on past information $\underline{b}_t = \underline{b}_t(\underline{x}_1, \ldots, \underline{x}_{t-1})$ is said to be *universal* if the wealth $S_T = \prod_{t=1}^{T} \underline{x}_t \cdot \underline{b}_t$ is guaranteed to be close to $\max_{\underline{b}} S_T(\underline{b})$. This guarantee can either through bounds that hold uniformly for all return sequences as in the work of Cover or through bounds that depend on additional properties of the returns as in our analysis.

3.2 Our Mixture Portfolio Analysis

In this section we build a mixture of portfolios involving subsets of all stocks, with weights determined by wealth achieved by these subset portfolios. Let $i_1, \ldots, i_k$ be the indices of a subcollection of the M stocks in which repeats are allowed. There are M^k such ordered subcollection and our strategy distributes wealth (initially equally) across all of these subcollection. For each ordered subcollection we provide portfolios weights to which these assets are rebalanced. The resulting wealth $S_{T,k}^{mix}$ after T investment periods is obtained by adding up the multi-period contribution from each subcollection.

3.2.1 Our Strategy and Result

One may think of there being a portfolio manager for each of the subcollection of stocks, each of whom is contracted to follow a prospectus specifying particular portfolio weights to which the stocks are to be rebalanced each period. Our wealth then is the sum of the wealths achieved by each of these managers, weighted by the (equal) fraction of our money initially placed in these funds. One may also think of the mixture as producing at the start of each period a portfolio depending only on the returns up to that time. Thus an alternative implementation is to compute that portfolio update each period and buy and sell as needed to achieve it.

We emphasize the distinction between the wealth $S_{T,k}^{mix}$ which is achievable (either as a mixture of funds or as an updating rule depending only on available return history each period) and the unachievable wealths S_T^k and S_T^{max} which we use as target wealths. The following theorem gives a sense in which $S_{T,k}^{mix}$ is near S_T^{max} for every reasonable return sequence provided the number of time periods T is large compared to the logarithm of the number of candidate stocks.

Theorem 2. *For our mixture strategy, there are choices of k of order $\sqrt{T/\log M}$ such that at time T we achieve a return $S_{T,k}^{mix}$ which has a wealth exponent that drops from S_T^{max} by not more than order $\sqrt{\frac{\log M}{T}}$. Specifically,*

$$S_{T,k}^{mix} \geq S_T^{max} e^{-T(a\sqrt{\frac{\log M}{T}})} \tag{3.1}$$

where $a = 2c$ and c is the function of stock return relative volatility specified in Theorem 1 of Chapter 2.

If we have prior knowledge of the value of c determined by the stock return relative volatility, we could set $k = c\sqrt{T/\log M}$ which would optimize our bound on the drop to be $2c\sqrt{(\log M)/T}$ where c is as given in Theorem 1. Prior knowledge of the value of c is generally not available. Thus we may use $k = \sqrt{T/\log M}$, which also leads to the same order bound (albeit with a dependence on volatility with c^2 in place of c).

Alternatively we may adapt to what is achieved by the best k, by distributing our initial wealth according to a prior $q(k)$ on the subcollection size $k = 1, \ldots, M$. That is $S_T^{mix} = \sum_{k=1}^{M} q(k) S_{T,k}^{mix}$. For example when $q(k)$ equals $1/M$, we distribute initial wealth evenly across all k.

Theorem 3. *For our mixture strategy, we achieve a return S_T^{mix} which has a wealth exponent that drops from S_T^{max} by not more than order $\sqrt{\frac{\log M}{T}}$ for arbitrary length of investment period T. Specifically,*

$$S_T^{mix} \geq S_T^{max} e^{-T(a\sqrt{\frac{\log M}{T}})} \tag{3.2}$$

where $a = 2c$ and c is the function of stock return relative volatility specified in Theorem 1 of Chapter 2.

This strategy does not require knowledge of the number of investment period T and the return volatility v in advance. Both fixed k and the adaptive strategy are shown to provide the bound in the proofs of Theorem 2 and Theorem 3 which are given in Chapter 4. Strengthening of the conclusions is also given there.

The Cover and Ordentlich (1996) universal portfolio strategy (details see in Section 3.3) achieves a wealth exponent that is within

$$\frac{M-1}{2T}\log\frac{T}{2\pi}+\frac{c_M}{T} \tag{3.3}$$

of the maximum wealth exponent y^{max}. Compared to our mixture, their universal portfolio achieves an exponent closer to y^{max} for T large compared to M. However, an often more realistic setting has M large compared to T, but T large compared to $\log M$. In this case, our mixture strategy drop $\sqrt{\frac{\log M}{T}}$ is smaller. A slight refinement of expression (3.3) is minimax optimal (as shown in Xie and Barron (2000)) where in the minimax formulations the maximum is taken over all possible return vectors. Indeed we emphasize that the Cover and Ordentlich (and the Xie and Barron) portfolio strategies achieve the bound on the drop in wealth exponent relative to the maximum uniformly over all return sequences. Our improvement (in which the M is replaced by a $\log M$) is not uniform over all return vectors but rather it depends on the observed volatility. Our analysis is related in that we also use a mixture based portfolio. The difference is that our π is discrete with points on faces of the simplex determined by subsets of stocks.

Having outlined above our conclusions for mixture portfolios, we turn in the next section to develop this theory and strategy in further detail. We will show experiments in Chapter 5 based on our mixture strategies with real stock market data.

3.2.2 Additional Detail Concerning Our Mixture Strategies

As we have written in previous section, we allocate our wealth across all subgroups of size k, which are constructed by selecting k stocks $(j_1, \ldots, j_k)$ from total M stocks. Hence there will be a total of M^k subgroups with size k when repeating is allowed. We will see later that allowing repeats of stocks in each subgroup will provide some freedom in the weights of each selected stocks. For each subgroup $(j_1, \ldots, j_k)$, the fractions of wealth on each stock follow the vector $\underline{\alpha}_k$ with the form of (2.7) which is the k-step portfolio in our wealth maximization algorithm. Here, we set portfolio weight for the most recent selected stock (k^{th} stock) as

$$\alpha_k = \frac{2}{k+2}. \tag{3.4}$$

Although the fractions of wealth on stocks of each ordered subgroup are fixed, the repeats of stocks in each subgroup can provide the freedom in weights for different stocks by adjusting the weight for the stock that has been selected already.

Let $S_T^{(j_1,\ldots,j_k)}$ be one of such subgroup wealth associated with the choice of k stocks $(j_1, \ldots, j_k)$ where the portfolio is determined by the weight vector $\underline{\alpha}_k$ given as following

$$\underline{\alpha}_k = \left(\frac{6}{(k+1)(k+2)}, \frac{6}{(k+1)(k+2)}, \frac{8}{(k+1)(k+2)}, \ldots, \frac{2}{k+2} \right). \tag{3.5}$$

Here the coordinates follow the pattern $\alpha_{i,k} = \frac{2(i+1)}{(k+1)(k+2)}$ for $i = 2, \ldots, k$. These are the weights that arise by initializing $\alpha_{1,1} = 1$ and then for $k > 1$ obtaining $\underline{\alpha}_k$ by multiplying $\underline{\alpha}_{k-1}$ by $\frac{k}{k+2}$ and setting the new coordinate to $\frac{2}{k+2}$. For example with $k = 5$, the portfolio vector $\underline{\alpha}_5$ for each subgroup with five stocks is

$$\left(\frac{3}{21}, \frac{3}{21}, \frac{4}{21}, \frac{5}{21}, \frac{6}{21} \right).$$

These weights give more attention to recent iterations than earlier ones. [In contrast if $\underline{\alpha}_k$ were formed by multiplying $\underline{\alpha}_{k-1}$ by $\frac{k-1}{k}$ and setting the new coordinates to $\frac{1}{k}$ then all k coordinates of $\underline{\alpha}_k$ would be equal.] The assignment of $\underline{\alpha}_k$ might seem mysterious now, we will discuss the reason in the following.

In our mixture strategies, we create two types of mixtures, one in which k is prespecified and another in which we mix across all possible k.

For the prespecified k case, we give equal initial allocation $1/M^k$ on each subgroup. Thus after T investment periods, we will have the wealth $S_T^{mix,k}$, where

$$S_T^{mix,k} = \frac{1}{M^k} \sum_{(j_1,\ldots,j_k)\in\{1,\ldots,M\}} S_T^{(j_1,\ldots,j_k)}.$$

To explain the idea of this mixture we contrast it with one particular subset stocks with wealth after T investment period $S_T^k = S_T^{(j_1^*,\cdots,j_k^*)}$ where $(j_1^*,\cdots,j_k^*)$ are the indices of the stocks selected the first k steps in our wealth maximization algorithm.

Therefore we set our portfolio vector $\underline{\alpha}_k$ equal to the first k step fraction of wealth of the algorithm. That choice of $j_1^*, \cdots, j_k^*$ depends on the entire return sequence $\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_T$, so it is impossible for the real investment to catch that choice before we start investment at time 1. However, our mixture overcomes the lack of advanced knowledge of which choice will perform well for the entire investment periods by giving some weight $(1/M^k)$ to every $(j_1, \ldots, j_k)$ at the starting of investment. By assigning $\underline{\alpha}_k$, our wealth maximization algorithm insures us before time 1 that we will have at least one subgroup $(j_1^*, \cdots, j_k^*)$ that will achieve a wealth exponent of more than

$$S_T^{max} e^{-T\frac{c^2}{k}}. \tag{3.6}$$

at the end of period T. Accordingly we find that

$$S_T^{mix,k} \geq \frac{1}{M^k} S_T^k.$$

However, for this prespecified k, it is still not strong enough to show our mixture strategy is a universal portfolio strategy because one need to know the number of investment periods T in advance. For real investment, people usually do not have any ideas of how long the investment will exist before closing or when they need to cash out the money from the market. Fortunately, this can be solved by using our second type of mixture in which we mix across all possible size k of stock subgroups. As k is the number of stocks chosen from all candidate M stocks and included into our portfolio $\underline{b}$, the largest possible k is M that is to include all stocks into the portfolio which means that our choices of k have the range of 1 to k. This type of mixture will be sufficient for our strategy to a universal one.

To mixture across all k, we need to assign a prior $q(k)$ as a weight to each k, for example one can choose uniformly prior $1/M$ by equally splitting our initial wealth to all k. Other distributions are also possible, especially when people have some prior knowledge, it is helpful to put more weight on some k. After giving weight to each k, we have total M sub-portfolios indexed by k, each of which is applied our first type of mixture by assigning weight $(1/M^k)$ to every $(j_1, \ldots, j_k)$. Therefore, we have the form of our wealth factor function S_T^{mix}, where

$$S_T^{mix} = \sum_{k=1}^{M} \frac{1}{M^{k+1}} \sum_{(j_1,\ldots,j_k)\in\{1,\ldots,M\}} S_T^{(j_1,\ldots,j_k)}.$$

In the next two sections we discuss Cover's universal portfolios and Hembold et al's exponential gradient learning portfolios.

3.3 Bayesian Estimation and Cover's Universal Portfolios

As we shall mention there are variations on the universal portfolio idea. Here we follow Cover in that the target level of performance is $S_T^{max} = \max_{\underline{b}} S_T(\underline{b})$ associated with the best constant rebalanced portfolio. Cover (1990) (as refined in Cover and Ordentlich (1996)) gives a particular mixture based strategy that we will explain further for which the wealth realized satisfies $S_T \geq S_T^{max} e^{-T\triangle_{M,T}}$ where the drop from the best exponent is bounded by $\triangle_{M,T} = \frac{1}{T}(\frac{M-1}{T} \log T + C_M)$ uniformly over all $\underline{x}_1, \ldots, \underline{x}_T$. As we have said, bounds of this type are best possible if the guarantee is to be for all return sequences. In contrast we have developed a new portfolio strategy that has a better bound than Cover's for return sequences having a volatility constraint and to simply the computation of the mixture.

As we do not have the knowledge of stock prices at time t until the end of t, it is impossible to invest directly with the best constant rebalanced portfolio $\underline{b}^{max}$. However surprisingly, Cover (1991) and Cover & Ordentlich's (1996) universal strategies would achieve S_T^{max} to the first order in exponent. This portfolio sequence starts by equally splitting wealth into all stocks and for the successive time periods, wealth is realocated according to $\hat{\underline{b}}_t$, where

$$\hat{\underline{b}}_t = \frac{\int \underline{b} S_{t-1}(\underline{b})\pi(\underline{b})d\underline{b}}{\int S_{t-1}(\underline{b})\pi(\underline{b})d\underline{b}} \tag{3.7}$$

for $t = 2, \ldots, T$ and where $\pi(\underline{b})d\underline{b}$ is a density function on the $(M-1)$-dimensional simplex of all possible portfolios $\underline{b}$.

We also can regard the inner product of portfolio $\underline{b}$ and stock return vector $\underline{x}_t$ at time t as the likelihood of $\underline{b}$ (for more on the likelihood interpretation see Chapter 6). Accordingly, $S_T(\underline{b})$ becomes the joint likelihood of $\underline{b}$. Hence given a prior distribution $\pi(\underline{b})$ and stock return sequences $\underline{x}_1, \ldots, \underline{x}_{t-1}$, the posterior of $\underline{b}$ is $\pi(\underline{b}|\underline{x}_1, \ldots, \underline{x}_{t-1})$ with

$$\pi(\underline{b}|\underline{x}_1, \ldots, \underline{x}_{t-1}) = \frac{S_{t-1}(\underline{b})\pi(\underline{b})}{\int S_{t-1}(\underline{b})\pi(\underline{b})d\underline{b}}. \tag{3.8}$$

Now we can see that the Bayes estimator $\hat{\underline{b}}_t^{Bayes}$ actually equals Cover's universal portfolio for time t, that is

$$\hat{\underline{b}}_t^{Bayes} = \frac{\int \underline{b} S_{t-1}(\underline{b})\pi(\underline{b})d\underline{b}}{\int S_{t-1}(\underline{b})\pi(\underline{b})d\underline{b}} \tag{3.9}$$

We discuss the stochastic model further in Chapter 6. The universal portfolios do not rely on stochastic assumptions.

A remarkable property Cover shows in (1991) for these sequences of portfolios is that the associated actual wealth $S_T^{univ} = \prod_{t=1}^T \hat{\underline{b}}_t \cdot \underline{x}_t$ also equals

$$\int S_T(\underline{b})\pi(\underline{b})d\underline{b} \tag{3.10}$$

for a continuous prior $\pi(\underline{b})$ and

$$\sum_{\underline{b}} S_T(\underline{b})\pi(\underline{b}) \tag{3.11}$$

for a discrete prior $\pi(\underline{b})$.

Proof:

$$\begin{aligned} S_T^{unif} &= \prod_{t=1}^T \hat{\underline{b}}_t \cdot \underline{x}_t \\ &= \prod_{t=1}^T \frac{\int \underline{b}\underline{x}_t S_{t-1}(\underline{b})\pi(\underline{b})d\underline{b}}{\int S_t(\underline{b}\pi(\underline{b})d\underline{b}} \\ &= \prod_{t=1}^T \frac{\int S_t(\underline{b})\pi(\underline{b})d\underline{b}}{\int S_{t-1}(\underline{b})\pi(\underline{b})d\underline{b}} \\ &= \int S_T(\underline{b})\pi(\underline{b})d\underline{b} \end{aligned}$$

Equivalently, one may think of π with $\int \pi(\underline{b})d\underline{b} = 1$ as providing an initial distribution of wealth over a continuum of investment managers, each of whom with a portfolio

$\underline{b}$, from whom we subsequently accumulate the total wealth $\int S_T(\underline{b})\pi(\underline{b})d\underline{b}$ at the end of time T.

The wealth S_T^{Bayes} associated with $\hat{\underline{b}}_t^{Bayes}$ is same as the wealth of Cover's universal portfolios, that is

$$S_T^{Bayes} = \int S_T(\underline{b})\pi(\underline{b})d\underline{b} \tag{3.12}$$

for a continuous prior and

$$S_T^{Bayes} = \sum_{\underline{b}} S_T(\underline{b})\pi(\underline{b}) \tag{3.13}$$

for a discrete prior case.

3.4 Hembold et al's Exponential Gradient Learning

Helmbold, et al. (1998) show a similar exponent drop of order $\sqrt{(\log M)/T}$ when using the following exponential gradient (EG) portfolio updating rule $b_{t+1,i}$ at time t with learning rate $\eta = 2c'\sqrt{2(\log M)/T}$ where

$$b_{t+1,i} = \frac{b_{t,i}\exp(\eta x_{t,i}/\underline{b}_t \cdot \underline{x}_t)}{\sum_{j=1}^{M} b_{t,j}\exp(\eta x_{t,j}/\underline{b}_t \cdot \underline{x}_t)}.$$

Here $c' = \min x_{t,i}$ for all $t \geq 1$ and $i \geq 1$. However, for EG learning the choice of proper η requires advance knowledge of both the price relative volatility bound c' and the number of trading periods T.

An refinement of EG learning shown to be universal for all possible return is also provided in Helmbold et al. (1998) by using the following portfolio update algorithm which is parameterized by a real number $\alpha \in [0, 1]$. Let

$$\underline{\tilde{x}}_t = (1 - \alpha/M)\underline{x}_t + \frac{\alpha}{M} 1_M$$

where 1_M is the vector with all coordinates equal one. The new EG learning update is

$$b_{t+1,i} = \frac{b_{t,i} \exp(\eta \tilde{x}_{t,i} / \underline{b}_t \cdot \underline{\tilde{x}}_t)}{\sum_{j=1}^{M} b_{t,j} \exp(\eta \tilde{x}_{t,j} / \underline{b}_t \cdot \underline{\tilde{x}}_t)}.$$

The associated wealth drop bound has the rate of $((M^2 \log M)/T)^{1/4}$ for $T \geq 2M^2 \log M$. The primary advantage of EG learning update is that it is easier to compute than Cover's.

In the next Chapter we will show proofs and further analysis on our wealth maximization theory and mixture strategy portfolios.

Chapter 4

Proof of Our Theorems

In this chapter, we will prove our main results summarized in Chapter 2 and Chapter 3. In Section 1 we establish the wealth maximization theorem after some preliminary lemmas.

In Section 2, we establish the mixture strategy theorem for universal portfolio of the prespecified stock subsets of size k and also prove Theorem 3 for investing on stock subsets mixing across all k, for $k = 1, \ldots, M$.

4.1 Proof of the Wealth Maximization Bound

We define D_k as the average logarithm ratio of $\underline{b}^{max} \cdot \underline{x}_t$ and $\underline{b}_k \cdot \underline{x}_t$ for $t = 1, \ldots, T$. That is

$$D_k = \frac{1}{T} \sum_{t=1}^{T} \log \frac{\underline{b}^{max} \cdot \underline{x}_t}{\underline{b}_k \cdot \underline{x}_t} = \frac{1}{T} \log \frac{S_T^{max}}{S_T(\underline{b}_k)}. \tag{4.1}$$

Theorem 1 states a bound on D_k of $c^2/(k+3)$. We prove this theorem through the following lemmas.

Lemma 1. *Suppose a sequence of nonnegative numbers D_k, with $k \geq 1$, satisfies*

$$D_k \leq (1-\alpha)D_{k-1} + \alpha^2 c^2/4 \tag{4.2}$$

for all $\alpha \in (0,1)$ and $k \geq 2$, for some c which is independent of k. Also suppose $D_1 \leq c^2/4$, then we have for all $k \geq 1$

$$D_k \leq \frac{c^2}{k+3}. \tag{4.3}$$

Proof: We proceed by induction. First, the bound holds by assumption when $k = 1$. Now suppose

$$D_{k-1} \leq \frac{c^2}{k+2}$$

for $k \geq 2$. Then invoking (4.2) with

$$\alpha = \frac{2}{k+2}$$

we have the following result

$$\begin{aligned} D_k &\leq (1 - \frac{2}{k+2})\frac{c^2}{k+2} + \frac{c^2}{(k+2)^2} \\ &= c^2 \cdot \left[\frac{k}{(k+2)^2} + \frac{1}{(k+2)^2}\right] \\ &= \frac{c^2}{k+3} \cdot \frac{k^2+4k+3}{(k+2)^2} \\ &\leq \frac{c^2}{k+3}. \end{aligned}$$

Though the statement of Lemma 1 requires the inequality (4.3) to hold for all $\alpha \in (0,1)$, we see from the proof that having (4.3) hold for $\alpha \leq 1/2$ and indeed for the particular choice $\alpha_k = 2/(k+2)$ with $k = 2, 3, \ldots$ is sufficient for the validity of the claim.

To show Theorem 1 from Lemma 1, we prove that D_k defined as in expression (4.1) indeed satisfies the requirement of inequality (4.2). Demonstration of this property of D_k is the focus of our remaining efforts in this section.

Here we need some useful inequalities for pairs of nonnegative real numbers.

Lemma 2. *For all numbers $r, r_0 \in R^+$ with $r_0 \leq r, r \neq 1$ and $r_0 \neq 1$, we have the inequality,*

$$-\log r \leq -(r-1) + [\frac{-\log r_0 + r_0 - 1}{(r_0 - 1)^2}](r-1)^2. \tag{4.4}$$

which means the function

$$f(r) = \frac{-\log r + r - 1}{(r-1)^2} \tag{4.5}$$

is monotone decreasing with $r \in R^+$.

Proof It is natural to check with the first derivative $f'(r)$ of the target function, where

$$f'(r) = \frac{1}{r-1}\left[\frac{1}{r} - \frac{2}{r-1} + \frac{2\log r}{(r-1)^2}\right]. \tag{4.6}$$

Obviously, $f'(r) < 0$ when $0 < r < 1$. What we need to show is $f'(r) < 0$ for $r > 1$, which is equivalent to show

$$\frac{1}{r} < \frac{2(r-1) - 2\log r}{(r-1)^2} \tag{4.7}$$

for $r > 1$. In order to show (4.7), we just need to show $\tilde{f}(r) = r\log r - \frac{r^2-1}{2} < 0$, which can be seen easily by checking the negative first derivative of $\tilde{f}(r)$. Actually,

$f(r)$ approaches 0.5 when r comes close to 1, although we have not define the value at $r = 1$. Hence $f(r)$ is a monotone decreasing function for $r \in R$ and $r \neq 1$.

Lemma 3. *For $r \in R^+$ and $r \neq 1$, the following inequality holds*

$$2[\frac{-\log r + r - 1}{r-1}] \leq \log r.$$

Proof: This is straightforward by showing a non-negative function $f(r)$ where

$$f(r) = \frac{1}{2}\log r - \frac{-\log r + r - 1}{r-1}.$$

What we need to show is that for $r > 1$ and $0 < r < 1$, the first derivative $f'(r)$ is always positive.

Lemma 4. *For $r \in R^+$ and $r \neq 1$, we have,*

$$\frac{-\log r + r - 1}{(r-1)^2} \leq 1/2 + \max(0, -\log r).$$

Proof For $r > 1$, under Lemma 3 and the fact $\log r \leq r - 1$, the inequality is proved. We just need to show it is also true for $0 < r < 1$. That is to prove for $0 < r < 1$, the following inequality holds

$$\frac{-\log r + r - 1}{(r-1)^2} \leq 1/2 - \log r.$$

It is equivalent to show for $0 < r < 1$

$$f(r) = \frac{1}{2}(r-1)^2 - (r-1)^2 \log r + \log r - r + 1 \geq 0 \tag{4.8}$$

Hence when

$$f'(r) = r - 1 - \frac{(r-1)^2}{r} - 2(r-1)\log r + \frac{1}{r} - 1 < 0 \tag{4.9}$$

saying the monotonicity of $f(r)$, (4.8) will be sufficed as $f(1) = 0$. Similarly, because $f'(1) = 0$, it is sufficient to show $f''(r) = 2/r - 2\log r - 2 > 0$, which is obvious under the fact $\log r \leq r - 1$ for all $r \in R^+$. Hence we have shown the Lemma 4.

Remark: Li and Barron (2000) use the same inequalities as above but with different settings in their work on mixture density estimation. They showed that for mixture density estimation, a k-component mixture estimated by maximum likelihood achieves log-likelihood within order $1/k$ of the log likelihood achievable by any convex combination.

For our analysis, we define r_t as a ratio of our portfolio return to the optimal portfolio return at time t at step k when stock i is introduced. That is,

$$r_t = \frac{(1-\alpha)Z_{t,k-1} + \alpha x_{t,i}}{\underline{b}^{max} \cdot \underline{x}_t}.$$

Also let $r_{0,t} = \frac{(1-\alpha)Z_{t,k}}{\underline{b}^{max} \cdot \underline{x}_t}$ where $0 < r_{0,t} \leq r_t, 0 \leq \alpha \leq 1$ and $Z_{t,k-1}$ and $\underline{b}^{max}$ are defined the same as in the previous section. Now we can start to show our main result. Plug r_t and $r_{0,t}$ into (4.4) and use (4.5) to obtain

$$\begin{aligned}
& -\log r_t \\
\leq\ & -(r_t - 1) + [\frac{-\log(r_{0,t}) + r_{0,t} - 1}{(r_{0,t}-1)^2}](r_{0,t} - 1 + \frac{\alpha x_{t,i}}{\underline{b}^{max} \cdot \underline{x}_t})^2 \\
=\ & -(r_{0,t} - 1 + \frac{\alpha x_{t,i}}{\underline{b}^{max} \cdot \underline{x}_t}) + (-\log r_{0,t} + r_{0,t} - 1) \\
& + (\frac{\alpha x_{t,i}}{\underline{b}^{max} \cdot \underline{x}_t})^2 (\frac{-\log r_{0,t} + r_{0,t} - 1}{(r_{0,} - 1)^2}) + \frac{2\alpha x_{t,i}}{\underline{b}^{max} \cdot \underline{x}_t}(\frac{-\log r_{0,t} + r_{0,t} - 1}{r_{0,t} - 1}) \\
\leq\ & -\log r_{0,t} - \frac{\alpha x_{t,i}}{\underline{b}^{max} \cdot \underline{x}_t} + (\frac{\alpha x_{t,i}}{\underline{b}^{max} \cdot \underline{x}_t})^2 (\frac{-\log r_{0,t} + r_{0,t} - 1}{(r_{0,t} - 1)^2}) \\
& + \frac{\alpha x_{t,i}}{\underline{b}^{max} \cdot \underline{x}_t} \log r_{0,t} \qquad (4.10)
\end{aligned}$$

This $-\log r_t$ appears in our update rule for D_k. Indeed, by the definition of D_k (equation (4.1)) with i_k is chosen to maximize expression (2.5). We have that

$$D_k \le \min_i D_{k,i} \tag{4.11}$$

where

$$D_{k,i} = \frac{1}{T}\sum_{t=1}^{T}\left[-\log \frac{(1-\alpha)Z_{t,k-1} + \alpha x_{t,i}}{\underline{b}^{max}\cdot \underline{x}_t}\right]. \tag{4.12}$$

This minimum is not less than the weighted average of $D_{k,i}$ for any weights that add to 1. In particular, the minimum is smaller than the average using $\underline{b}^{max}$. Hence a sequence of inequalities can be given as follows,

$$\begin{aligned}
D_k &\le \sum_{i=1}^{M} b_i^{max} D_{k,i} \\
&= \frac{1}{T}\sum_{t=1}^{T}\sum_{i=1}^{M} b_i^{max}\left[-\log r_{0,t} - \frac{\alpha x_{t,i}}{\underline{b}^{max}\cdot \underline{x}_t}\right. \\
&\quad \left. +\left(\frac{\alpha x_{t,i}}{\underline{b}^{max}\cdot \underline{x}_t}\right)^2\left(\frac{-\log r_{0,t} + r_{0,t} - 1}{(r_{0,t}-1)^2}\right) + \frac{\alpha x_{t,i}}{\underline{b}^{max}\cdot \underline{x}_t}\log r_{0,t}\right] \\
&= -\log(1-\alpha) - \alpha + \alpha\log(1-\alpha) + (1-\alpha)D_{k-1} \\
&\quad +\frac{\alpha^2}{T}\sum_{t=1}^{T}\sum_{i=1}^{M} b_i^{max}\left(\frac{x_{t,i}}{\underline{b}^{max}\cdot \underline{x}_t}\right)^2\left(\frac{-\log r_{0,t} + r_{0,t} - 1}{(r_{0,t}-1)^2}\right) \\
&\le (1-\alpha)D_{k-1} + \frac{\alpha^2}{T}\sum_{t=1}^{T}\sum_{i=1}^{M} b_i^{max}\left(\frac{x_{t,i}}{\underline{b}^{max}\cdot \underline{x}_t}\right)^2\left(\frac{-\log r_{0,t} + r_{0,t} - 1}{(r_{0,t}-1)^2}\right)
\end{aligned}$$

where the last inequality is established by noting $(\alpha-1)\log(1-\alpha)-\alpha \le 0$ for $\alpha \in [0,1]$.

Write

$$I = \frac{1}{T}\sum_{t=1}^{T}\sum_{i=1}^{M} b_i^{max}\left(\frac{x_{t,i}}{\underline{b}^{max}\cdot \underline{x}_t}\right)^2 \tag{4.13}$$

and

$$v = \max_{1 \le t \le T, 1 \le i,j \le M} \{x_{t,i}/x_{t,j}\}. \tag{4.14}$$

Let

$$x_{t,max} = \max_i \{x_{t,i}\}$$

and

$$x_{t,min} = \min_i \{x_{t,i}\}. \tag{4.15}$$

Since $\sum_{i=1}^{M} b_i^{max} x_{t,i} \le x_{t,max}$, likewise $Z_{t,k-1} \ge x_{t,min}$, we have by Lemma 4 that

$$\begin{aligned} & \frac{-\log r_{0,t} + r_{0,t} - 1}{(r_{0,t} - 1)^2} \\ \le \; & 1/2 + \log^{-}(r_{0,t}) \\ = \; & 1/2 + \log^{+}\left[\frac{\sum_{i=1}^{M} b_i^{max} x_{t,i}}{(1-\alpha)\sum_{i=1}^{M} b_{i,k-1} x_{t,i}}\right] \\ \le \; & 1/2 + \log 2v \\ = \; & \log 2v\sqrt{e} \end{aligned}$$

Thus the inductive inequality $D_k \le (1-\alpha)D_{k-1} + \alpha^2 c^2/4$ is obtained, where,

$$c^2 = 4I \log 2v\sqrt{e}. \tag{4.16}$$

As mentioned before, D_k is the normalized log-wealth ratio, so we conclude that

$$D_k = \frac{1}{T} \log \frac{S_T^{max}}{S_T^k} \le \frac{c^2}{k+3}. \tag{4.17}$$

That is,

$$S_T^k \ge S_T^{max} e^{-T\frac{c^2}{k+3}}. \tag{4.18}$$

So we showed the proof of Theorem 1, which says that the computation of S_T^k approximates S_T^{max} with an exponent that is less than the maximum by not more than $c^2/(k+3)$.

Regarding the quantities I and v we note that

$$\begin{aligned} I &= \frac{1}{T}\sum_{t=1}^{T}\sum_{i=1}^{M} b_i^{max}\left(\frac{x_{t,i}}{\underline{b}^{max}\cdot \underline{x}_t}\right)^2 \\ &= \frac{1}{T}\sum_{t=1}^{T}\left(\frac{\sum_{i=1}^{M} b_i^{max} x_{t,i}}{\sum_{i=1}^{M} b_i^{max} x_{t,i}}\cdot x_{t,i}/\sum_{i=1}^{M} b_i^{max} x_{t,i}\right) \\ &\leq \frac{1}{T}\sum_{t=1}^{T}\max_{1\leq i\leq M}\left(\frac{x_{t,i}}{\underline{b}^{max}\cdot \underline{x}_t}\right) \end{aligned} \tag{4.19}$$

which in term is not more than v. Thus I is the mildest of these volatility expressions depending only on relative return of stocks in the portfolio $\underline{b}^{max}$ relative to the portfolio return $\underline{b}^{max}\cdot\underline{x}_t$. In contrast the bound (4.18) depends on the maximum relative return over all stocks (though still relative to $\underline{b}^{max}\cdot\underline{x}_t$) and the measure of volatility v depends on the worst case ratio overall.

4.2 Proof of Two Mixture Strategy Theorems

In Chapter 3, we have discussed our mixture strategy for universal portfolios by allocating our wealth across all stock subgroups of fixed size k. We also discussed about rather than the prespecified k, it is preferable to mix across all choices of k. Here we need to emphasize the difference between our mixture strategy wealth $S_{T,k}^{mix}$ or S_T^{mix} and the results by using our wealth maximizing algorithm S_T^k. The former two are achievable which is only depending on available information and the latter is unachievable for which is what we targeted.

We first show Theorem 2 for fixed k. That is, we mix all portfolios with k stocks which might be repeated to allow freedom in the weights. We need be cautious that here k is required to be integers, however, the optimal k to satisfy the following theorems might not be integers. We will show it is also true under the integer requirement for both theorems.

Proof of Theorem 2: Among the subgroups we mix across it will happen that one of them will be the particular one $(j_1^*, \ldots, j_k^*)$ that arises by our greedy algorithm in Theorem 1. Then invoking our bound on its wealth we have

$$\begin{aligned}
S_T^{mix,k} &= \frac{1}{M^k} \sum_{(j_1,\ldots,j_k)\in\{1,\ldots,M\}} S_T^{(j_1,\ldots,j_k)} \\
&\geq \frac{1}{M^k} \max_{(j_1,\ldots,j_k)} S_T^{(j_1,\ldots,j_k)} \\
&\geq S_T^{(j_1^*,\ldots,j_k^*)} \frac{1}{M^k} \\
&\geq S_T^{max} e^{-T\left[\frac{c^2}{k} + \frac{k \log M}{T}\right]} \\
&\geq S_T^{max} e^{-T \cdot 2c\sqrt{\frac{\log M}{T}}} \qquad (4.20) \\
&= e^{T\left[y^{max} - 2c\sqrt{\frac{\log M}{T}}\right]}.
\end{aligned}$$

Here to achieve the bound in (4.19) one sets $k = c\sqrt{\frac{T}{\log M}}$ which optimizes the bound on the drop to be $2c\sqrt{(\log M)/T}$, where c is given in Theorem 1. When prior knowledge of c is not available, we may use $k = \sqrt{T/\log M}$, which also results a bound of the same order. That is,

$$\begin{aligned} S_T^{mix,k} &\geq S_T^{max} e^{-T(c^2(v)+1)\sqrt{\frac{\log M}{T}}} \\ &= e^{T\left[y^{max}-(c^2(v)+1)\sqrt{\frac{\log M}{T}}\right]} \end{aligned}$$

This completes our proof of Theorem 2.

Rather than using a fixed size k, when c is unknown, it is preferable to mix across choices of k. We may distribute our initial wealth according to a prior $q(k)$ on the subcollection size k for $k = 1, \ldots, M$. This mixing procedure leads to wealth S_T^{mix} after T investment periods, where

$$S_T^{mix} = \sum_{k=1}^{M} q(k) S_{T,k}^{mix}.$$

Particularly, we can take a uniform prior $q(k) = \frac{1}{M}$ yielding

$$S_T^{mix} = \sum_{k=1}^{M} \frac{1}{M^{k+1}} \sum_{j_1,\ldots,j_k \in \{1,\ldots,M\}} S_T^{(j_1,\ldots,j_k)} \tag{4.21}$$

Proof of Theorem 3: As shown in Section 2.1, we have the inequality

$$\frac{1}{T} \log \frac{S_T^{max}}{S_T^k} \leq \frac{c^2}{k+3}.$$

Thus

$$\begin{aligned} &\frac{1}{T} \log \frac{S_T^{max}}{S_T^k} \\ \leq\ & \frac{c^2}{k+3} \\ \leq\ & \frac{c^2}{k+1}. \end{aligned}$$

Equivalently, $S_T^k \geq S_T^{max} e^{-T\frac{c^2}{k+1}}$. Hence using the same argument as above we have,

$$\begin{aligned}
S_T^{mix} &\geq \sum_{k=1}^{M} \frac{1}{M^{k+1}} S_T^{max} e^{-T\frac{c^2}{k+1}} \\
&= \sum_{k=1}^{M} S_T^{max} e^{-T[\frac{c^2}{k+1} + \frac{(k+1)\log M}{T}]} \\
&\geq S_T^{max} e^{-T \cdot 2c\sqrt{\frac{\log M}{T}}} \qquad (4.22) \\
&= e^{T\left[y^{max} - 2c\sqrt{\frac{\log M}{T}}\right]}.
\end{aligned}$$

This complete our proof of Theorem 3.

We notice that we have not used the requirement of k to be integers in the above proof or Theorem 2 and Theorem 3. However, as k is the number of stocks in a subset, it has to be an integer. We will show that Theorem 2 and Theorem 3 will not only been satisfied by this requirement, but also our lower bound can be improved by multiplying M^2 and M, respectively. First we need the following lemma.

Lemma 5. *For $k \in N$ and $\kappa \in R^+$, we have an inequality*

$$\min_{k \in N}\{\frac{c^2}{k+3} + \frac{k \log M}{T}\} \leq \min_{\kappa \in R^+}\{\frac{c^2}{\kappa+2} + \frac{\kappa \log M}{T}\} \qquad (4.23)$$

where $c^2(v), M$ and T have the same definitions as the above.

Proof: We know the righthand side of (4.23) can achieve the minimum by

$$\kappa^* = c(v)\sqrt{T/\log M} - 2 \qquad (4.24)$$

Let an integer k_1 be such that $k_1 \leq \kappa^* + 2 < k_1 + 1$. We first assume

$$\frac{c^2}{k_1} + \frac{k_1 \log M}{T} \leq \frac{c^2}{k_1+1} + \frac{(k_1+1)\log M}{T}. \qquad (4.25)$$

This means that the right side of (4.23) is smaller with k_1, the first smaller integer than $\kappa^* + 2$, than with $k_1 + 1$. obviously, (4.25) implies that

$$\frac{\log M}{T} \geq \frac{c^2}{k_1} - \frac{c^2}{k_1 + 1}. \tag{4.26}$$

Therefore,

$$\begin{aligned}
& \min_{k \in N}\{\frac{c^2}{k+3} + \frac{k \log M}{T}\} \\
= \; & \frac{c^2}{k_1} + \frac{k_1 \log M}{T} - \frac{3 \log M}{T} \\
\leq \; & \frac{(\kappa^* + 2) \log M}{T} + c^2 \left[\frac{1}{k_1} + \frac{1}{k_1 + 1} - \frac{1}{k_1}\right] - 2\frac{\log M}{T} \\
\leq \; & \frac{(\kappa^* + 2) \log M}{T} + \frac{c^2}{(\kappa^* + 2)} - \frac{2 \log M}{T} \\
= \; & \min_{\kappa \in R^+}\{\frac{c^2}{\kappa + 2} + \frac{\kappa \log M}{T}\}.
\end{aligned}$$

The case where the right side of (4.23) is smallest at $k_1 + 1$ is handled similarly. This complete the proof of Lemma 5. We apply it to the Theorem 2 and Theorem 3 in the following.

Corollary 1. *Under the same condition of Theorem 2, we have the following strengthened inequality*

$$S_T^{mix,k} \geq M^2 e^{T\left[y^{max} - 2c\sqrt{\frac{\log M}{T}}\right]}. \tag{4.27}$$

Proof: The analysis is similar to the proof of Theorem 2. Indeed,

$$\begin{aligned}
S_T^{mix,k} & \geq S_T^{(j_1^*, \ldots, j_k^*)} \frac{1}{M^k} \\
& \geq S_T^{max} e^{-T\left[\min_{k \in N}\left(\frac{c^2}{k+3} + \frac{k \log M}{T}\right)\right]} \\
& \geq S_T^{max} e^{-T\left[\min_{\kappa \in R^+}\left(\frac{c^2}{\kappa+2} + \frac{\kappa \log M}{T}\right)\right]}
\end{aligned}$$

$$
\begin{aligned}
&\geq S_T^{max} e^{-T\cdot 2c\sqrt{\frac{\log M}{T}}+2\log M} \\
&= M^2 e^{T\left[y^{max}-2c\sqrt{\frac{\log M}{T}}\right]}.
\end{aligned}
$$

Here the real number κ^* minimizes the exponent equals $c\sqrt{T/\log M}-2$. Hence our choice of integer k^* equals either k_1-3 or k_1-2 with $k_1 \leq \kappa^* < k_1+1$ depending on which one has the smaller value in the function

$$\frac{c^2}{k}+\frac{k\log M}{T}.$$

Corollary 2. *Under the same condition of Theorem 3, we have the following strengthened inequality*

$$S_T^{mix} \geq M e^{T\left[y^{max}-2c\sqrt{\frac{\log M}{T}}\right]}. \tag{4.28}$$

Proof: Similarly,

$$
\begin{aligned}
S_T^{mix} &\geq \sum_{k=1}^{M} \frac{1}{M^{k+1}} S_T^{max} e^{-T\frac{c^2}{k+3}} \\
&= \sum_{k=1}^{M} S_T^{max} e^{-T\left[\frac{c^2}{k+3}+\frac{(k+1)\log M}{T}\right]} \\
&= \sum_{k=1}^{M} S_T^{max} e^{-T\left[\min_{\kappa\in R^+}\left(\frac{c^2}{\kappa+2}+\frac{(\kappa+1)\log M}{T}\right)\right]} \\
&\geq S_T^{max} e^{-T\cdot 2c\sqrt{\frac{\log M}{T}}+\log M} \\
&= M e^{T\left[y^{max}-2c\sqrt{\frac{\log M}{T}}\right]}.
\end{aligned}
$$

This completes our proof of Corollary 2.

Hence, we have shown that under the integer requirement, the theorems remain true. Moreover, we have shown the improved lower bounds.

In the next chapter, we apply both our wealth maximization algorithm and the mixture strategy to real stock data from Wharton Research Data Services.

Chapter 5

Experiments with Real Market Data

In this chapter, we conduct several experiments with real stock data to examine historical stock performance with our wealth maximizing algorithm and to test the performance of our mixture portfolio strategy and some other practical strategies.

5.1 Mixture Strategy Experiments

For illustrative purposes, we examine investment in Standard and Poor 500 stocks. The stock price information is from the Wharton WRDS online data base. First we examine the consequence of following our mixture strategy over a 10 year period from January 1996 through December 2005. From Figure 1, we can see that our mixture portfolio multiplies wealth by the factor $S_T^{mix} = 6.10$ (e.g. a \$1000 initial investment

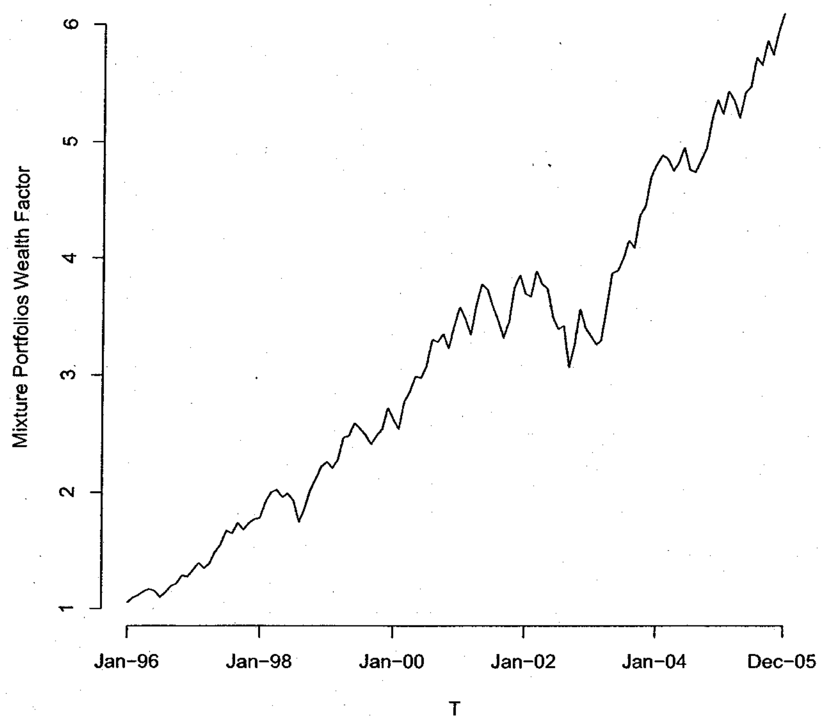

Figure 5.1: Performance of The Mixture Portfolio Strategy S_T^{mix} on S&P 500 for $T = 1$ to 120 months

would have become \$6,100). Here we are rebalancing monthly the portfolios of subsets of these stocks and aggregating them together at the end. Here we use all subsets of size $k = 3$. The mixture we form totals the wealth from all 500^3 subsets with repeats allowed. [Thus each of the subsets is given initial weight weight $\frac{1}{500^3}$]. The wealth factor 6.10 for these ten years coincides with an annual return of about 19.82%. The best subset stocks among all are *Dell Inc.*, *Jabil Circuit Inc.* and *Qlogic Corp.* which with weights $(0.3, 0.3, 0.4)$ had wealth factor $S_T = 125.1$ and the best individual company multiperiod return was 67.5 by *Jabil Circuit Inc.*.

Next we examine investment in Standard and Poor 500 stocks over the 26 years period from January 1970 through December 1995 with rebalancing monthly among subsets of these stocks. This time frame is chosen for comparison with result of Goetzmann (1996). Our mixture portfolio multiplies wealth by a factor of 34.4, which is equivalent to an annual compounding rate of return of 14.7%. Goetzmann reports a 12% annual return, which is the average return for these periods instead of the actual compounding rate, under the mean-variance criterion. As we know, the arithmetic mean is always greater than the geometric mean. That is,

$$\begin{aligned} \frac{1}{T}\sum_{t=1}^{T} \underline{b}\cdot\underline{x}_t &\geq (\prod_{t=1}^{T}\underline{b}\cdot\underline{x}_t)^{\frac{1}{T}} \\ &= e^{y(\underline{b})} \\ &= 1+r(\underline{b}). \end{aligned}$$

Thus the actual compounding rate of return is less than the empirical average return. In particular, for Goetzmann's strategy the actual compounding rate of return which is not reported must be not more than 12% per year whereas our rate of 14.7% is higher. In our work we use only those stocks listed in S&P 500 for the entire period from January 1970 through December 1995. It is not clear how Goetzmann handles those stocks delisted from S&P 500 during this period.

A critic might complain that maximization of $y(\underline{b}) = \sum_{t=1}^{T}\log(\underline{b}\cdot\underline{x}_t)$ does not have a variance constraint. Nevertheless we hasten to point out that the average logarithm (as arise in maximization of $S_T(\underline{b})$) is a more risk adverse criterion especially for $\underline{b}\cdot\underline{x}_t$

near 0 than quadratic utilities. Such risk adversion in $y(\underline{b})$ is necessary for identifying the highest attainable rate of growth for constant rebalanced portfolios.

Also we point out that the 14.7% return is the return attained by the mixture strategy which can be regarded as updated each period based only on preceding performance. In contrast the 12% average return given by Goetzmann is based on hindsight for the whole 26 year period. We turn attention next to what would be the best growth rate with hindsight.

5.2 Maximum Wealth Index Calculation

In this section, we first compute the maximum constant rebalanced portfolio wealth factor $S_T^{max} = \max_{\underline{b}} S_T(\underline{b})$ for the ten year period from January 1996 through December 2005, rebalancing monthly (that is for $T = 120$ months). We use as the pool of stocks those that have been included as Standard and Poor 500 stocks with their monthly return as reported by the Wharton data base. We find that with 3 or 4 stocks the greedy algorithm comes reasonably close to the maximum.

k	Stocks in Portfolio at Each Algorithm Step	Wealth Factor S_T^k
1	BIIB	155.97
2	BIIB, CTXS	942.90
3	BIIB, CTXS, NTAP	1585.51
4	BIIB, CTXS, NTAP	1606.96
5	BIIB, CTXS, NTAP	1613.13
6	BIIB, CTXS, NTAP	1615.14
7	BIIB, CTXS, NTAP, APOL	1616.54
8	BIIB, CTXS, NTAP, APOL	1617.17
.	...	.
.	...	.
.	...	.
14	BIIB, CTXS, NTAP, APOL	1618.31
15	BIIB, CTXS, NTAP, APOL, DELL	1618.38
16	BIIB, CTXS, NTAP, APOL, DELL	1618.39
.	...	.
.	...	.
.	...	.
1000	BIIB, CTXS, NTAP, APOL, DELL	1618.41

Table 1: Wealth Maximization Algorithm

Indeed from Table 1, we see that the algorithm only needs $k = 4$ with three stocks to achieve a wealth factor S_T^k of 1606.96. Further optimization, for instance to $k = 16$ steps reaches a factor S_T^k of 1618.39 with only five stocks. These stocks are *Biogen Idec Inc. (BIIB), Citrix Systems Inc. (CTXS), Network Appliance Inc. (NTAP), Apoppo Group Inc. (APOL)* and *Dell Inc. (DELL)*. In this implementation

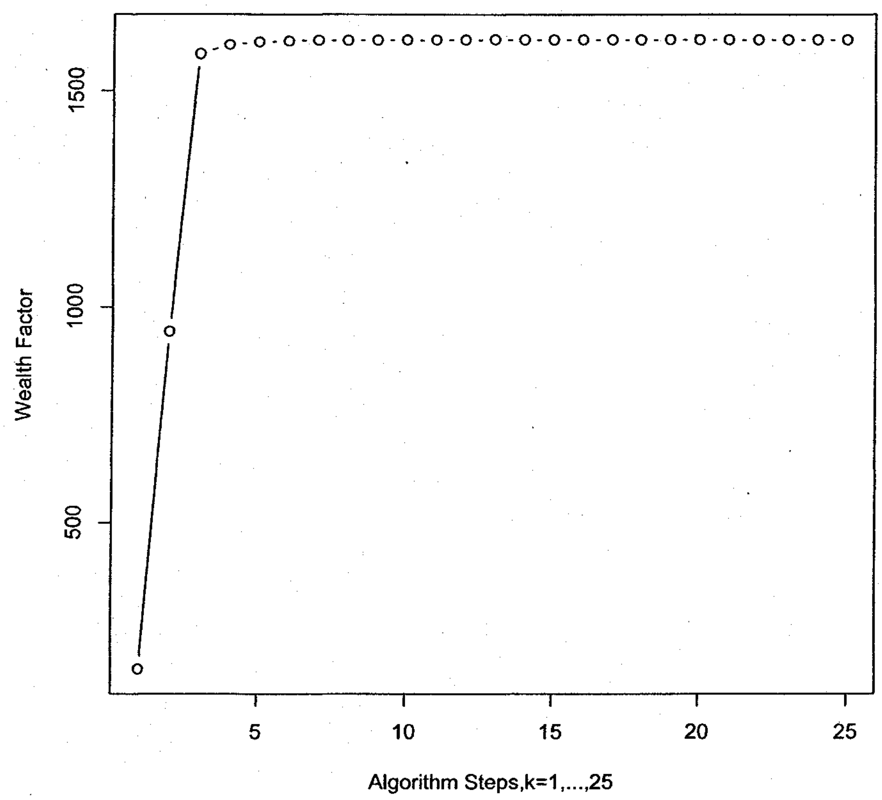

Figure 5.2: Twenty-year maximum rebalanced wealth approximation S_T^k for $k = 1, 2, \ldots, 25$ by greedy algorithm on all S&P 500 stocks

for $k > 1$ we allowed α_k to be freely adjusted between 0 and 1/2 (here we used a fine grid of spacing 1/10000) rather than fixed at $\alpha_k = \frac{2}{k+2}$ (either way is permitted by our theory). Each step tries every stock for possible new inclusion or tunes an existing stock weight (relative to the others), whichever is best. For these data, the algorithm found no advantage after step 15 for including any additional stocks beyond the indicated five. Thus confirming that we were already very close to the maximum with a handful of steps.

Figure 2 plots the increasing wealth factor S_T^k at each step and Figure 3 shows the corresponding drop from the maximal exponent. The volatility quantities I and $\log(v)$ in this experiment are 1.04 and 5.97 respectively. This wealth growth of $S_T^{max} = 1618.4$ over the $T = 120$ months reflects a monthly wealth exponent of $y^{max} = \frac{1}{120} \log 1618.4 = 0.0615$ or equivalently an annualized return of $(1618.4)^{\frac{1}{10}} = 2.09$, that is, 109% growth per year. Though surprisingly high, we must emphasize that the tradeoffs required for achievable wealth are sobering, even in this example. Indeed consider the mixture wealth bound from Theorem 2. With $M = 500$ stocks, $\sqrt{(\log M)/T} = \sqrt{0.05} = 0.22$ so even if c^2 were near $I = 1.04$ the bound on the drop in exponent swallows the otherwise spectacular gain of y^{max}.

We also computed the maximum constant rebalanced portfolio wealth with monthly rebalancing for the twenty year period from January 1986 through December 2005 for Standard and Poor 500 stocks. At step $k = 4$ it uses the fours stocks, *Apple Inc. (AAPL), Countrywide Financial Corp. (CFC), Roclwell Automation (ROK)* and *RadioShack Corp. (RSH)*, to achieve a wealth factor S_T^k of 1667.09. There are five more stocks, *MBNA Corp. (KRB), Safeway Inc. (SWY), Synovus Financial Corp. (SNV), American Bankers Ins Group Inc. (ABI)*, and *Keyspan Energy Corp.(KSE)*, when further optimized to 21 steps and the wealth factor after 1000 steps is 1829.52. The corresponding volatility quantities I and $\log(v)$ are 1.008 and 2.18 respectively.

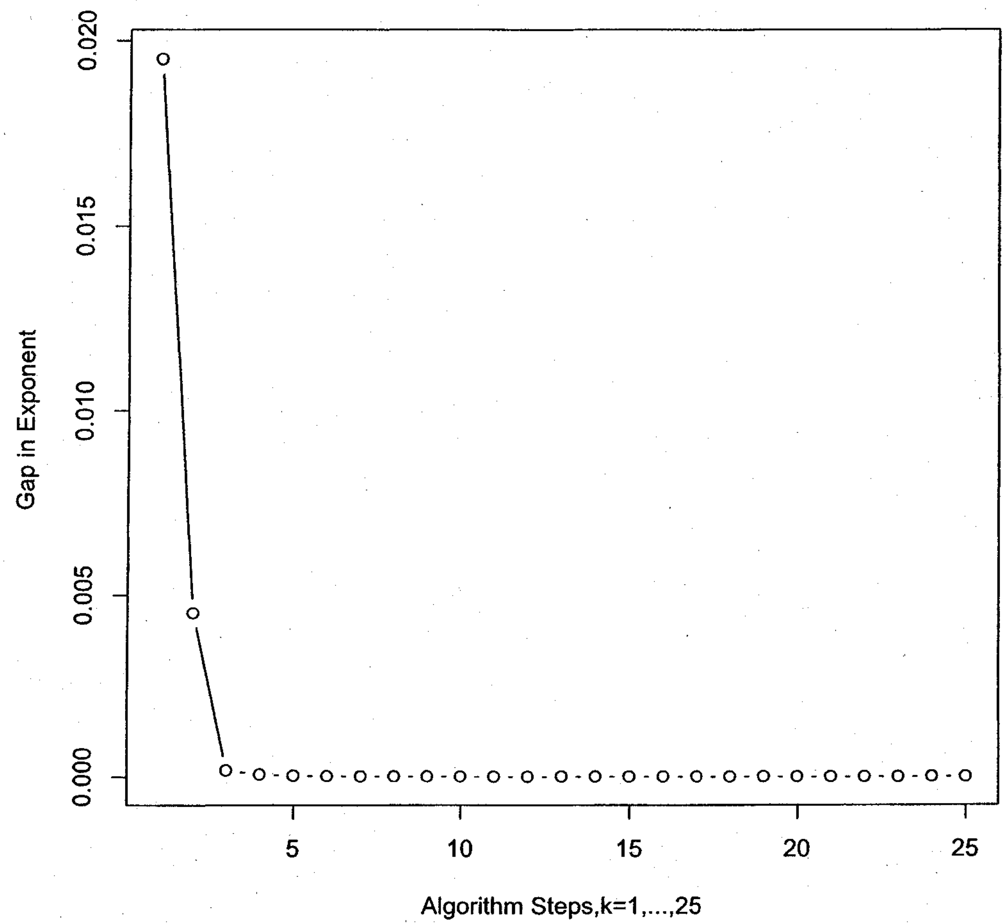

Figure 5.3: Gap in exponent between S_T^{max} and S_T^k for $k = 1, \ldots, 25$

Finally, we consider the period from January 1970 through December 1995 and Standard and Poor 500 stocks as in Goetzmann (1996). With monthly rebalancing the best constant rebalanced portfolios with hindsight achieved a multiperiod compounding wealth factor of 561.8 running the algorithm up to 500 steps. The corresponding annual compounding rate of return is 27.5%. Here we also exclude those stocks that were delisted during this period. At step $k = 7$, the portfolio has a wealth factor of 561.08 and uses the five stocks, *Mylan Labs Inc. (MYLN)*, *Southwest Airlines Co. (LUV)*, *St Jude Medical Inc. (STJM)*, *Home Depot Inc. (HD)* and *Circuit City Stores Inc. (CC)*.

5.2.1 Moving-Window Greedy Updating Versus the Mixture Strategy

Here we report a strategy of greedy portfolio selection using what may be called moving window information. In this strategy at each trading time (e.g. at the end of each month), the portfolio we set for the next period (e.g. the next month) is the portfolio which, on the previous T_w time periods, would have made the most wealth as computed by our greedy algorithm. We shift this time window (for training the next portfolio) each period so that it reflects the same window length T_w of past return. Our theory gives no guarantee that the next period behavior is predicted best by the preceding time window, nonetheless, it is of interest to see how such a greedy strategy would have performed. Our wealth maximization algorithm is the key ingredient for computation of this moving-window algorithm.

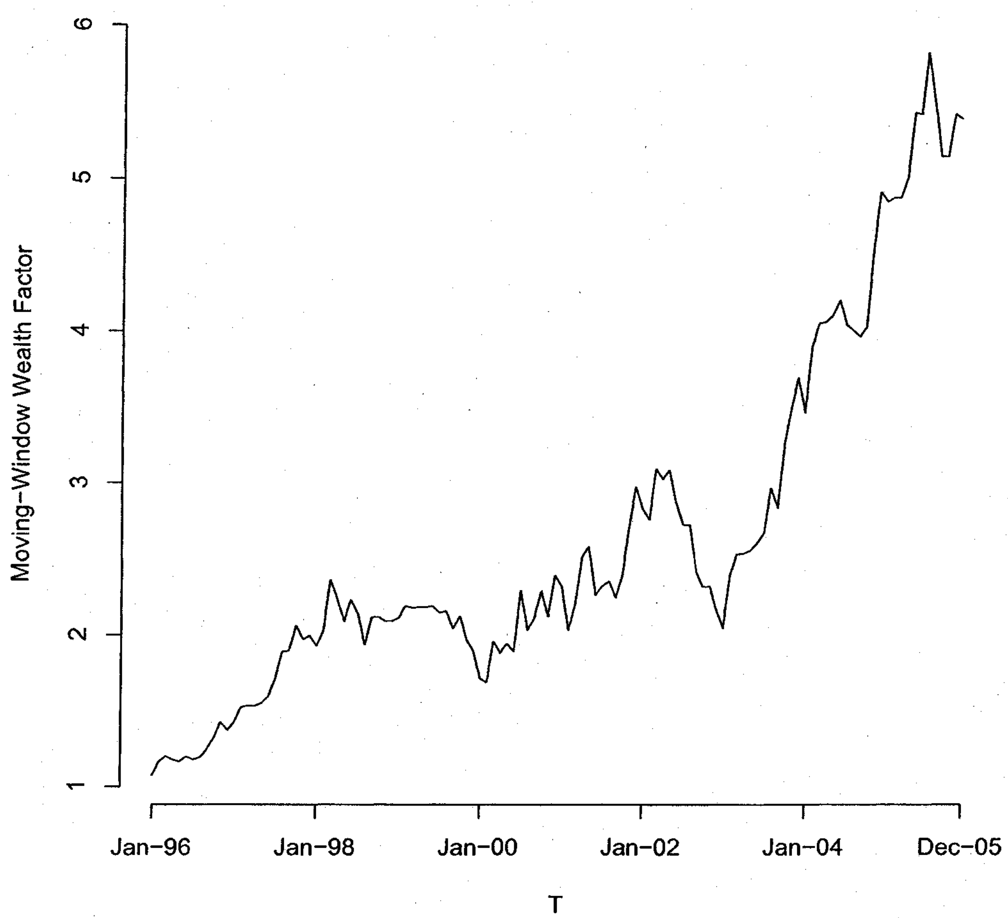

Figure 5.4: Wealth factor for moving-window portfolio updates

Here we again use the Standard and Poor 500 Index stocks. For each month from January 1996 through December 2005, we consider a moving training window of preceding returns with which we determine the portfolio to use for the current month. For the length of the training window, 12, 18, 24 and 30 months were tried. The best results as reported here were based on a 24 month window. There is also the issue of the rapidity of rebalancing for the portfolio wealth function that we maximize over the preceding years.

For instance, it is unclear whether it is better to use the stock fractions that are optimal with daily rebalancing or with monthly rebalancing. For ease of computation we report results in which we tried monthly rebalancing on each training window here.

Thus at the start of each month we get the portfolio which would (with monthly rebalancing) have made the most over the preceding two years and set that to be our portfolio for the start of that month. It is then updated (with a moving two years window) at the start of the following month. During these 10 years our portfolio has a wealth factor over 5.39 (or an annual return of 18.3%) as shown in Figure 3. Each month it used a small handful of stocks that evolved across time.

Another result shows a twenty-year result in which our moving-window portfolio achieves a wealth factor of 23.43 (or an annual return of 17%) during this period from January 1986 through December 2005 with a training window of 24 months for Standard and Poor 500 stocks while the S&P 500 index gained a factor of 5.89 (or a annual return of 9.3%) during that period.

We also applied this strategy to the NASDAQ 100 stocks over the January 1997 through December 2005 time frame with daily rebalacing on the training window. We tried $6, 9, 12, 15, 18, 21$ and 24 months as the length of training window. The best one is with 12 months window. It shows an impressive result that during these 9 yeas our portfolio has a wealth factor of 78.6 while the NASDAQ 100 index gained a factor of 2.14 in the same period. For a 24 months training window the wealth factor for these 9 years is about 36.8. It is still better than the result from Standard and Poor stocks. The NASDAQ companies are relative younger and faster growing companies with smaller capitalization compared with Standard and Poor companies, which might be one of the reasons for this phenomenon.

With the same stocks that we are using in the previous two sections from January 1970 through December 1995, the moving window wealth factor is 37.8 with monthly rebalancing on a twelve month training window. Equivalently, it is an annual return of 15% while during the same period the S&P 500 index has an annual return of 8%.

Thus the moving window greedy algorithm has performed quite well on recent historical data and should be given serious attention as an investment strategy. However, we caution that the greedy approach can be fooled. Sudden changes in best portfolios can lead to a situation in which a portfolio trained to be best on the past is miserable in the future (compared to appropriate targets).

As our Theorem 2 shows, the mixture strategy advocated in this paper does provides a performance guarantee. It is provably close in exponent to the best constant rebalanced portfolio (however high or low that might be) provided $c\sqrt{(\log M)/T}$ is small compared to that best exponent.

5.2.2 Wealth Targets with Past Dependent Portfolios

In this paper we have presented the algorithm in the context of optimization of constant rebalanced portfolios, where we find $\underline{b}$ such that $\Pi_{t=1}^{T}\underline{b} \cdot \underline{x}_t$ is maximized where each $x_{t,i}$ is the return of an available asset. However, it is possible to incorporate interesting types of past dependence in this framework. The idea is to allow parameterized dependence of the portfolio on past returns so as to capture the possibility of putting higher or lower attention on stocks that went up the previous period. Cross & Barron (2003) introduce such past dependent portfolio in a universal portfolio setting. As in [14], if the dependence of the portfolios on past returns is linear in portfolio weights then our theory readily adapts to this setting.

In particular consider vectors such as

$$\underline{\tilde{x}}_t^+ = (\frac{(x_{t,1}-1)_+}{s_t^+}, \ldots, \frac{(x_{t,M}-1)_+}{s_t^+}) \tag{5.1}$$

and

$$\underline{\tilde{x}}_t^- = (\frac{(x_{t,1}-1)_-}{s_t^-}, \ldots, \frac{(x_{t,M}-1)_-}{s_t^-}) \tag{5.2}$$

which are non-negative, sum to 1, and depend on $\underline{x}_t$. Here $s_t^+ = \sum_{i=1}^{M}(x_{t,i}-1)_+$ and $s_t^- = \sum_{i=1}^{M}(x_{t,i}-1)_-$. Then for period t, the above vectors $\underline{\tilde{x}}_{t-1}^+$ or $\underline{\tilde{x}}_{t-1}^-$ computed from the preceding period may be thought of as providing portfolios for new auxiliary assets $x_{t,M+1} = \underline{\tilde{x}}_{t-1}^+ \cdot \underline{x}_t$ and $x_{t,M+2} = \underline{\tilde{x}}_{t-1}^- \cdot \underline{x}_t$.

Now a portfolio $\underline{b} \cdot \underline{x}_t = \sum_{i=1}^{M+2} b_i x_{t,i}$ in all the assets (including the two newly created) may be regarded as investing in each stock i a fraction $b_i + b_{M+1}\tilde{x}_{t-1,i}^+ +$

$b_{M+2}\tilde{x}^{-}_{t-1,i}$ which depends on the past and is indeed linear in the weight $\underline{b}$. Including this freedom for past dependence (captured through the auxiliary assets) the wealth target $\max_{\underline{b}} \Pi_{t=1}^{T} \underline{b} \cdot \underline{x}_t$ is now higher than before. It is still available for computation by our wealth maximization algorithm and for construction of mixture-based or moving window portfolio updates.

For example, if each month we put weight 1/2 on $\underline{\tilde{x}}^{-}_{t-1}$ and weight 1/2 on the moving-window greedy portfolio updates described before, then the nine-year NASDAQ 100 wealth factor ending December 2005 increases from 78.6 to 146.1.

Chapter 6

Stochastic Perspective of Optimal Portfolios

We have already known that our wealth maximization theory and mixture strategies do not reply on any stochastic assumptions. In this chapter, we will discuss some portfolios estimations from the stochastic perspective.

6.1 Stochastic Maximal Wealth Portfolios

An "Efficient" market hypothesis is the assumption that stock return sequences $\underline{X}_1, \underline{X}_2, \cdots, \underline{X}_T$ are i.i.d. with some know probability distribution P. It is natural to consider the maximization of $E_P \log S_T(\underline{b})$ in order to maximize T period wealth factor S_T with all return sequences are i.i.d. under distribution P. We then call $\underline{b}^*$ the portfolio that maximizes $E_P \log S_T(\underline{b})$ with $\underline{b}^* = (b_1^*, b_2^*, \cdots, b_M^*)$.

We define Q to be the empirical distribution with mass function

$$q(\underline{x}) = \frac{1}{T}\sum_{t=1}^{T} 1\{\underline{x}_t = \underline{x}\}$$

and define $Q_{\underline{b}_k,\underline{b}^{max}}$ to be the wealth drop distribution with mass function

$$q_{\underline{b},\underline{b}^{max}} = \frac{\underline{b}\cdot\underline{X}}{\underline{b}^{max}\cdot\underline{X}}q(\underline{X}).$$

By the Law of Large Numbers we can see that

$$\lim_{T\to\infty}\frac{1}{T}\log S_T(\underline{b}) = E_P \log S_T(\underline{b}).$$

Thus $S_T(\underline{b})$ is asymptotically maximized if we choose the constant rebalanced portfolio $\underline{b}^*$ to maximize $E_Q \log \underline{b}\cdot\underline{x}$. We effectively achieve the asymptotic optimal wealth factor. Moreover, Breiman (1961) shows the following inequality holds almost surely

$$\lim_{T\to\infty}\sup_{\{\underline{x}_t\}_{t=1}^{T}}\frac{1}{T}\log\frac{S_T(\underline{b})}{S_T(\underline{b}^*)} \leq 0 \tag{6.1}$$

where $S_T(\underline{b}^*)$ is the wealth factor achieved by maximizing $E_P \log \underline{b}\cdot\underline{x}$ and $S_T(\underline{b})$ is the wealth factor achieved by any other portfolio sequences.

Similar results by Bell and Cover (1980,1988) and Algoet and Cover (1988) are achieved with a weak condition that the stock return sequences do not need to satisfy the i.i.d. condition. The largest exponent $y(\underline{b})$ for stationary ergodic $\underline{X}$ and competitively largest in general occurs almost surely with $\underline{b}_t(past)$ chosen to achieve

$$\max_{\underline{b}} E_P[\log \underline{X}_t\cdot\underline{b}|past]. \tag{6.2}$$

Cover & Ordentlich (1996) and Cross & Barron (2003) allowed portfolios with past dependence or even more various of sources of side information, which can be past stock prices, economic indicators, analyst opinions, etc.). We can use parameterized portfolios to represent these situations, i.e. $\underline{b}_t = \underline{b}_t(\underline{\theta}, \underline{X}_1, \ldots, \underline{X}_{t-1})$ where $\underline{\theta}$ is a vector of side information variables. Thus the wealth factor function at time T is

$$S_T(\theta) = \prod_{t=1}^{T} \underline{b}_t(\underline{\theta}, \underline{X}_1, \ldots, \underline{X}_{t-1}) \cdot \underline{X}_t.$$

For the CRP, the portfolios for each period remain the same and only depend on $\underline{\theta}$, that is $\underline{b}_t = \underline{b} = \underline{\theta}$ for $t = 1, \ldots, T$.

We name a stock return distribution on X to be fair if $E[X] = 1$ and to be subfair if $E[X] \leq 1$, which means you keep your money in your pocket. Bell and Cover (1980, 1988) showed that $\underline{b}^*$ achieves $\max E_P[\log \underline{b} \cdot \underline{X}]$ if and only if

$$p^*(\underline{X}) = \frac{1}{\underline{b}^{max} \cdot \underline{X}} p(\underline{x}) \tag{6.3}$$

is a fair distribution that is

$$E_{P^*} X_i = 1 \tag{6.4}$$

for each i with $b_i^* > 0$ and is subfair that is

$$E_{P^*} X_i \leq 1 \tag{6.5}$$

for all assets i. By conditioning on past,

$$p^*(\underline{X}_t | \underline{X}_1, \ldots, \underline{X}_{t-1}) = \frac{1}{\underline{b}_t^*(\underline{X}_t | \underline{X}_1, \ldots, \underline{X}_{t-1}) \cdot \underline{X}_t} P(\underline{X}_t | \underline{X}_1, \ldots, \underline{X}_{t-1}) \tag{6.6}$$

provides a fair process. It makes $\{X_{t,i}\}$ a martingale for each stock i that remains active with $b_i^* > 0$ and all return sequences supermartingales.

Therefore, for any fair distribution P_0 there is linear statistical families $p(\underline{X}|\underline{b})$ where $p(\underline{X}|\underline{b}) = \underline{b} \cdot \underline{X} p_0(\underline{X})$. For the i.i.d. case, the likelihood function become

$$p(\underline{X}_1, \ldots, \underline{X}_T|\underline{b}) = \left(\prod_{t=1}^{T} \underline{b} \cdot \underline{X}_t \right) p_0(\underline{X}_1, \ldots, \underline{X}_T). \tag{6.7}$$

With the best constant rebalanced portfolio $\underline{b}$, the likelihood (6.8) is maximized. The family always contains the true distribution $P(\underline{X})$ by choosing $p_0(\underline{x}) = p^*(\underline{X}) = \frac{1}{\underline{b}^* \cdot \underline{X}} p(\underline{X})$ with portfolio $\underline{b}$ equals $\underline{b}^*$ as

$$p(\underline{X}|\underline{b}) = \underline{b} \cdot \underline{X} p^*(\underline{X}) = \frac{\underline{b} \cdot \underline{X}}{\underline{b}^* \cdot \underline{X}} p(\underline{X}).$$

We might regard $S_T(\underline{b})$ as the joint likelihood of $\underline{b}$ for $\underline{X}_1, \ldots, \underline{X}_T$. As in Section (3.1.1), we obtain the Bayes estimator $\hat{\underline{b}}_t^{Bayes}$ with a prior $\pi(\underline{b})$. Meanwhile, the optimal portfolio $\underline{b}^*$ is also the maximum likelihood estimator obtained by $\max_{\underline{b}} \log S_T(\underline{b})$.

Let us denote $p_{\underline{b},\underline{b}^*} = \frac{\underline{b} \cdot \underline{X}}{\underline{b}^* \cdot \underline{X}} p(\underline{X})$. Moreover, we can interpret $E_P \log \frac{\underline{b}^* \cdot \underline{X}_t}{\underline{b} \cdot \underline{X}_t}$, which is the expected difference for time t return, as a Kullback-Leibler divergence between $\underline{b}^{max} \cdot \underline{X}_t$ and $\underline{b}_k \cdot \underline{X}_t$, that is

$$D(P||P_{\underline{b},\underline{b}^*}) = E_P \log \frac{\underline{b}^* \cdot \underline{X}_t}{\underline{b} \cdot \underline{X}_t}$$

In Chapter 4, $\{D_k\}_{k=1}^{+\infty}$ can also been seen as a sequence of Kullback-Leibler divergence under the empirical distribution where

$$\begin{aligned} D_k &= \frac{1}{T} \log \frac{S_T^{max}}{S_T^k} \\ &= \frac{1}{T} \sum_{t=1}^{T} \log \frac{\underline{b}^{max} \cdot \underline{X}_t}{\underline{b}_k \cdot \underline{X}_t} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{\underline{X}} q(\underline{X}) \log \frac{q(\underline{X})}{\frac{\underline{b}_k \cdot \underline{X}}{\underline{b}^{max} \cdot \underline{X}} q(\underline{X})} \\
&= D(Q||Q_{\underline{b}_k, \underline{b}^{max}}).
\end{aligned}
$$

The (4.3) implies the logarithmic likelihood of $\underline{b}_k$ converges to the maximum logarithmic likelihood with drop from the maximum not more than order $1/k$.

We have the following lemma.

Lemma 6. *Let $D(P||P_{\underline{b},\underline{b}^*})$ defines as above and $D(\underline{b}^*||\underline{b})$ defined as following*

$$
D(\underline{b}^*||\underline{b}) = \sum_{i=1}^{M} b_i^* \log \frac{b_i^*}{b_i} \tag{6.8}
$$

where $\underline{b}$ is any portfolio vector. The following inequality holds

$$
D(P||P_{\underline{b},\underline{b}^*}) \leq D(\underline{b}^*||\underline{b}) \tag{6.9}
$$

Proof: Jensen's inequality implies that

$$
\begin{aligned}
& \log \frac{\underline{b} \cdot \underline{X}_t}{\underline{b}^* \cdot \underline{X}_t} \\
= \ & \log \left(\sum_{i=1}^{M} \frac{b_i^* X_{t,i}}{\underline{b}^* \cdot \underline{X}_t} \cdot \frac{b_i}{b_i^*} \right) \\
\geq \ & \sum_{i=1}^{M} \frac{b_i^* X_{t,i}}{\underline{b}^* \cdot \underline{X}_t} \log \frac{b_i}{b_i^*}.
\end{aligned}
$$

Using the fact (6.6) and averaging the above inequality with respect to P yields that

$$
\begin{aligned}
& \sum_{\underline{X}_t} p(\underline{X}_t) \log \frac{\underline{b} \cdot \underline{X}_t}{\underline{b}^* \cdot \underline{X}_t} \\
\geq \ & \sum_{i=1}^{M} b_i^* \log \frac{b_i}{b_i^*} \sum_{\underline{X}_t} \frac{p(\underline{X}_t) X_{t,i}}{\underline{b}^* \cdot \underline{X}_t}
\end{aligned}
$$

$$= \sum_{i=1}^{M} b_i^* \log \frac{b_i}{b_i^*}$$

Hence, $D(P||P_{\underline{b},\underline{b}^*}) \leq D(\underline{b}^*||\underline{b})$.

In the next section we define a portfolio distance d^2 and bound the portfolio risk of applying our maximization portfolio for historical data to future data under the assumption that the historical stock returns and future returns are either i.i.d. across time or independent while having different distributions.

6.2 Portfolio Risks

Assume the stock return vectors $\{\underline{x}_t\}_{t=1}^{T}$ are i.i.d with distribution P and density p. Denote $\mathcal{C}$ to be set of all possible portfolio $\underline{b}$ with $\mathcal{C} = \{\underline{b} : \sum b_i = 1, b_i \geq 0\}$. We will first establish a portfolio risk bound by defining a "portfolio distance" $d^2(\underline{b}, \underline{b}^*)$ under the assumption that the past stock return vectors, we also call "training data" or "old data", have identical distribution with future unknown return vectors which we also call "future data" or "new data". Then we will extend the result to the case that past returns and future unknown returns have different distributions, P and Q, respectively.

Let is define a portfolio distance $d^2(\underline{b}, \underline{b}^*)$ as the following

$$d^2(\underline{b}, \underline{b}^*) = E_P((\frac{\underline{b} \cdot \underline{X}}{\underline{b}^* \cdot \underline{X}})^{1/2} - 1)^2 \tag{6.10}$$

where $\underline{b}$ and $\underline{b}^*$ are an arbitrary portfolio and optimal portfolio with hindsight, respectively. Here $\underline{X}$ is return vector for any time t as they are i.i.d. across time

$t = 1, \ldots, T$.

We first establish the following inequality.

Lemma 7. *We have the following affinity bound for two portfolios $\underline{b}$ and $\underline{b}^*$,*

$$d^2(b, b^*) \leq 2 \log \frac{1}{E_P(\frac{b \cdot X}{b^* \cdot X})^{1/2}} \tag{6.11}$$

Proof: Applying the fact that $\log x \leq x - 1$ for $x > 0$, we can prove the lemma as following

$$\begin{aligned} & E_P((\frac{b \cdot X}{b^* \cdot X})^{1/2} - 1)^2 \\ = \; & 2(1 - E_P(\frac{b \cdot X}{b^* \cdot X})^{1/2}) \\ \leq \; & -2 \log E_P(\frac{b \cdot X}{b^* \cdot X})^{1/2}. \end{aligned}$$

We have an immediate result by Lemma 6 and Lemma 7.

Corollary 3.

$$d^2(\underline{b}, \underline{b}^*) \leq D(\underline{b}^* || \underline{b}) \tag{6.12}$$

Proof:

From Lemma 6 and Lemma 7,

$$\begin{aligned} & d^2(\underline{b}, \underline{b}^*) \\ \leq \; & -2 \log E_P(\frac{\underline{b} \cdot \underline{x}_t}{\underline{b}^* \cdot \underline{x}_t})^{1/2} \\ \leq \; & -2 E_P \log(\frac{\underline{b} \cdot \underline{x}_t}{\underline{b}^* \cdot \underline{x}_t})^{1/2} \\ = \; & E_P \log(\frac{\underline{b} \cdot \underline{x}_t}{\underline{b}^* \cdot \underline{x}_t}) \end{aligned}$$

$$\leq \quad D(\underline{b}^*||\underline{b}).$$

We now write $\hat{b}_{old}$ as the k^{th} step portfolio from our algorithm for training data and b^*_{new} as the maximal portfolio for future data. We call $E_P d^2(\hat{b}_{old}, b^*_{new})$ to be the portfolio risk of using past nearly optimal portfolio $\hat{b}_{old}$, which is obtained by applying our wealth maximization algorithm on historical data, on the future data instead of optimal portfolio b^*_{new}. We have the following theorems.

Theorem 4. *Let $X_{old} = (\underline{X}_1, \ldots, \underline{X}_T)$ be historical stock return vectors, or training data, for T investment periods of M stocks and i.i.d. with distribution P. Also write X_{new} as future unknown return vectors, which are also i.i.d with distribution P. Write $\mathcal{C}$ be a finite set of portfolios $\underline{b}$, where $\mathcal{C} = \{\underline{b} : \sum b_i = 1, b_i \geq 0\}$. We now have a portfolio risk bound as the following*

$$E_P d^2(\hat{b}_{old}, b^*_{new}) \leq \frac{c_P^2}{k+3} + \frac{2\log \mathcal{M}}{T}. \tag{6.13}$$

where $c_P^2 = Ec^2(\underline{X})$, in which $c^2(\underline{X}) = 4I \log 2v\sqrt{e}$, and $\mathcal{M}$ is the cardinality of set $\mathcal{C}$.

Proof: By definition of $d^2(\hat{b}_{old}, b^*_{new})$, we can have

$$\begin{aligned}
& Td^2(\hat{b}_{old}, b^*_{new}) \\
\leq \; & 2T \log \frac{1}{E_P(\frac{\hat{b}\cdot X}{b^*\cdot X})^{1/2}} \\
= \; & 2 \log \frac{1}{(E_P(\frac{\hat{b}\cdot X}{b^*\cdot X})^{1/2})^T} \\
= \; & 2 \log \frac{1}{E_P(\frac{S_T(\hat{b}_{old})}{S_T(b^*_{new})})^{1/2}}
\end{aligned}$$

$$= 2\log[\frac{1}{E_P(\frac{S_T(\hat{b}_{old})}{S_T(b^*_{new})})^{1/2}} \cdot (\frac{S_T(\hat{b}_{old})}{S_T(b^*_{new})})^{1/2}] + \log\frac{S_T(b^*_{new})}{S_T(\hat{b}_{old})}$$

We take expected value for (6.16). For the first of (6.16),

$$\begin{aligned}
& 2E_P \log[\frac{1}{E_P(\frac{S_T(\hat{b}_{old})}{S_T(b^*_{new})})^{1/2}} \cdot (\frac{S_T(\hat{b}_{old})}{S_T(b^*_{new})})^{1/2}] \\
\leq\ & 2E_P \log[\sum_b \frac{1}{E_P(\frac{S_T(b)}{S_T(b^*_{new})})^{1/2}} \cdot (\frac{S_T(b)}{S_T(b^*_{new})})^{1/2}] \\
\leq\ & 2\log E_P[\sum_b \frac{1}{E_P(\frac{S_T(b)}{S_T(b^*_{new})})^{1/2}} \cdot (\frac{S_T(b)}{S_T(b^*_{new})})^{1/2}] \\
=\ & 2\log \mathcal{M}
\end{aligned}$$

For the second part of (6.16), we have

$$\begin{aligned}
& E_P \log\frac{S_T(b^*_{new})}{S_T(\hat{b}_{old})} \\
=\ & E_P \log\frac{S_T(b^*_{new})}{S_T(\hat{b}_{old})} \cdot \frac{S_T(b^*_{old})}{S_T(b^*_{old})} \\
=\ & E_P \log\frac{S_T(b^*_{new})}{S_T(b^*)} + E_P \log\frac{S_T(b^*)}{S_T(\hat{b}_{old})} \\
\leq\ & \log E_P \frac{S_T(b^*_{new})}{S_T(b^*)} + T \cdot \frac{c_P^2}{k+3} \\
=\ & T \cdot \frac{c_P^2}{k+3}
\end{aligned}$$

where the inequality (6.16) is derived from Theorem 1. Hence, we have shown the portfolio risk bound

$$E_P d^2(b^*_{new}, \hat{b}) \leq \frac{c_P^2}{k+3} + \frac{2\log\mathcal{M}}{T}.$$

Our assumption in the previous theorem is that both the historical and future data have the same distribution P. Now we give a more general result without this assumption. In fact, we solve the case that the training data has distribution P and future data has distribution Q which is not necessarily the same as P.

Lemma 8. *Let b^*_{new}, b^*_{old} and $\hat{b}_{old}$ be defined the same as above. Also define $\hat{b}_{new}$ as the k^{th} step portfolio from our algorithm using future data as if it is known. We have the following inequality*

$$\int \log \frac{S(b^*_{new})}{S(\hat{b}_{old})} dQ \le T \cdot \frac{c_Q^2}{k+3} + TD(Q||P). \tag{6.14}$$

where $c_Q^2 = Ec^2(\underline{X})$.

Proof:

$$\begin{aligned}
& E_Q \log \frac{S(b^*_{new})}{S(\hat{b}_{old})} \\
= & \int \log \frac{S(b^*_{new})}{S(\hat{b}_{old})} dQ \\
= & \int \log(\frac{S(b^*_{new})}{S(\hat{b}_{old})} \frac{p}{q} \frac{q}{p}) dQ \\
= & \int \log(\frac{S(b^*_{new})}{S(\hat{b}_{old})} \frac{p}{q}) dQ + \int \log \frac{q(X_1, \cdots, X_T)}{p(X_1, \cdots, X_T)} dQ \\
= & \int \left[\log \frac{S(b^*_{new})}{S(\hat{b}_{new})} + \log(\frac{S(\hat{b}_{new})}{S(\hat{b}_{old})} \frac{p}{q}) \right] dQ + TD(Q||P) \\
\le & \; E_Q \log \frac{S(b^*_{new})}{S(\hat{b}_{new})} + \log E_P \frac{S(b^*_{new})}{S(\hat{b}_{old})} + TD(Q||P) \\
\le & \; T \frac{c_Q^2}{k+3} + TD(Q||P)
\end{aligned}$$

Now let us show the result for the case when there are two different distributions P and Q on training data and new data respectively.

Theorem 5. *Suppose training data $X_{old} \sim P$ and future data $X_{new} \sim Q$. $\hat{b}_{old}, \hat{b}_{new}, b^*_{old}$ and b^*_{new} have the same definition as above. Then*

$$E_Q d^2(\hat{b}_{old}, b^*_{new}) \le 2 \frac{\log \mathcal{M}}{T} + D(Q||P) + \frac{c_Q^2}{k+3}. \tag{6.15}$$

Proof:

$$
\begin{aligned}
& Td^2(\hat{b}_{old}, b^*_{new}) \\
= \;& TE_Q((\frac{\hat{b}_{old} \cdot X_{new}}{b^*_{new} \cdot X_{new}})^{1/2} - 1)^2 \\
\leq \;& 2\log \frac{1}{\int (\frac{S(\hat{b}_{old})}{S(b^*_{new})})^{1/2} dQ} \\
= \;& 2\log \frac{1}{\int (\frac{S(\hat{b}_{old})}{S(b^*_{new})})^{1/2} dQ} \cdot (\frac{S(\hat{b}_{old})}{S(b^*_{new})})^{1/2} + \log \frac{S(b^*_{new})}{S(\hat{b}_{old})}
\end{aligned}
$$

By taking expected value in the last equality, we can get

$$
\begin{aligned}
& TE_Q d^2(\hat{b}_{old}, b^*_{new}) \\
\leq \;& 2E_Q \log \frac{1}{\int (\frac{S(\hat{b}_{old})}{S(b^*_{new})})^{1/2} dQ} \cdot (\frac{S(\hat{b}_{old})}{S(b^*_{new})})^{1/2} + E_Q \log \frac{S(b^*_{new})}{S(\hat{b}_{old})} \\
\leq \;& 2E_Q \log \sum_{b \in C} \frac{1}{\int (\frac{S(b)}{S(b^*_{new})})^{1/2} dQ} \cdot (\frac{S(b)}{S(b^*_{new})})^{1/2} + T\frac{c^2(v)}{k+3} + TD(Q\|P) \\
\leq \;& 2\log \mathcal{M} + T\frac{c_Q^2}{k+3} + TD(Q\|P).
\end{aligned}
$$

Hence we have shown the result

$$
E_Q d^2(\hat{b}_{old}, b^*_{new}) \leq 2\frac{\log \mathcal{M}}{T} + D(Q\|P) + \frac{c_Q^2}{k+3}. \tag{6.16}
$$

We can see that if the distribution Q of future returns is identical with the historical return distribution P, which is the assumption of Theorem 4, Theorem 4 is implied from Theorem 5.

Chapter 7

Portfolios of Options

In this chapter, we will discuss on the wealth of constant rebalanced portfolios of stocks and options. The chapter is collected from the thesis prospectus of Jianfeng Yu (2003) and a unpublished manuscript from Jianfeng Yu and Andrew R. Barron.

7.1 Gambling on Horse Races

We will relate our portfolios of options to the pure horse racing gambling scenario with total M horses. In this case the vector of a gambler's betting fractions $\underline{b} = (b_0, b_1, \ldots, b_M)$ plays the role of the portfolio, where b_i is the fraction of money gambled on horse i and $b_0 = 1 - \sum_{i=1}^{M} b_i$ is the fraction left in his pocket. Let the odds be c_i for 1 (these odds are also denoted as 1 for $p_i^* = 1/c_i$ or reported as $c_i - 1$ to 1), meaning that if horse i wins then the wealth gambled on that horse is multiplied by

c_i. Then after T races, the wealth factor takes the form

$$S_T(\underline{b}) = \prod_{t=1}^{T} (b_0 + c_{s_t} b_{s_t})$$

where s_t is the horse that wins race t, for $t = 1, 2, \ldots, T$. Having a positive fraction b_0 reserved for the pocket can be useful when the odds are such that p_i^* sums to more than 1, reflecting a track take. In a sufficiently regulation-free racing market, a no-arbitrage (no free money) argument shows that the odds must satisfy $\sum_{i=1}^{M} p_i^* = 1$, and whence there is no need for retaining wealth in the pocket as this riskless asset is realizable by a combination of bets on the horses. In that case the compounded wealth $S_T(\underline{b})$ takes an especially simple product form

$$S_T(\underline{b}) = \prod_{t=1}^{T} c_{s_t} b_{s_t}$$

which might be exactly rewritten as

$$S_T(\mathbf{b}) = e^{TD(q||p^*) - TD(q||\underline{b})} \tag{7.1}$$

where $D(q||\underline{b}) = \sum_{i=1}^{M} q_i \log(q_i/b_i)$ is the Kullback-Leibleri divergence and where q_i is the relative frequency with which horse i wins the T races. The divergence $D(q||\underline{b})$ is non-negative and it equals zero only when $\underline{b}$ equals q.

What is important in this gambling story is that the wealth identity (7.1) lays bare the roles of choices of the vector of betting fractions $\underline{b}$ and of the reciprocal odds p^* compared to the relative frequency vector q. The wealth is a product of two factors $TD(q||p^*)$ and $e^{-TD(q||\underline{b})}$. The first governs the impact of the choice of payoff odds and the second reveals the role of the choice of betting fraction $\underline{b}$. With hindsight the maximal wealth betting fraction is explicitly $\underline{b}^* = q$, with corresponding maximal wealth $S_T(\underline{b}^*) = \max S_T(\underline{b}) = e^{TD(q||p^*)} = S_T^{max}$. Indeed, any $\underline{b}$ other than q yields exponentially smaller wealth by the factor $e^{-TD(q||\underline{b})}$. The theory of universal portfolios is simplest in the gambling case and permits solution of estimated portfolios that exactly minimize the worst case drop from the maximal compounded wealth building on earlier work by Shtarkov (1988). Moreover, these minimax strategies achieve a wealth exponent close to the best without prior knowledge of q uniformly over all possible race outcomes.

The use of options with a sufficiently complete set of strike prices enables a dramatic simplification of the stock investment story, both for pricing and for the choice of portfolios and universal portfolio estimates.

In brief, a set of stock options with sufficiently many strike price levels completes the market for that stock to provide opportunity to gamble on the exact state of the stock return. This enables us to provide exact decomposition of the wealth in portfolios of options in terms of the corresponding betting fractions on state securities. A difference from pure gambling is that avoidance of arbitrage restricts the reciprocal odds p^* to those that make the stock return $\underline{x}$ be fair, in the sense that $E_{p^*}\underline{x} = 1$. Capturing this aspect leads naturally to a wealth decomposition into a product of three factors as revealed in Theorem .

Armed with this wealth representation for option investment we provide simple expression for the portfolio of maximum wealth in terms of the relative frequencies of the states of the return. Furthermore, for portfolio estimation these wealth identities with options provide opportunity to determine exact minmax universal portfolios (uniformly over all stock outcome sequences) and to provide explicit easily computed expressions for universal portfolios.

7.2 Wealth Decomposition

As discussed in the introduction, when gambling on M possible states with relative frequencies q_i, reciprocal odds p_i^* and betting fractions b_i, starting with 1 dollar, the compounded wealth after T gambling periods is

$$S_T^{gambling}(\underline{b}) = \prod_{t=1}^{T} b_{s_t} c_{s_t} = e^{TD(q||p^*)} e^{-TD(q||\underline{b})}$$

such that, with hindsight, the best arbitrage-free odds for a bookie are $p^* = q$, and, likewise, the best betting fractions are $\underline{b} = q$. Moreover, the difference in the Kullback-Leibler divergences $D(q||p^*)$ and $D(q||\underline{b})$ quantifies the rates of growth of the compounded wealth.

We will first give analogous fact for stock portfolios, followed in the next sections by our result for portfolios of a stock and options.

7.2.1 Wealth Identity

Each occurrence of a return vector $\underline{x}$ in the sequence $\underline{x}_1, \ldots, \underline{x}_T$ contributes $1/T$ to the empirical distribution $q(\underline{x})$. We give a convex constraint set $\mathcal{C}$ for portfolio $\underline{w}$ where $\mathcal{C} = \{w : w_i \geq 0, \sum_{i=1}^{M} w_i = 1\}$. Here we prohibit selling short.

Now, we characterize the wealth in terms of q and $\underline{b}$.

Theorem 6. *The multiperiod wealth factor of a constant rebalanced portfolio $\underline{b}$ for T periods with relative frequencies of return q is*

$$S_T(\underline{w}) = e^{Ty(\underline{w})}$$

with exponent

$$y(\underline{w}) = D(q||q_0) - D(q||q_{\underline{w},\underline{w}^*})$$

where

$$q_{\underline{w},\underline{w}^*}(\underline{x}) = \frac{\underline{w} \cdot \underline{x}}{\underline{w}^* \cdot \underline{x}} q(\underline{x})$$

is nonnegative and has sum not more than one

$$\sum_{\mathbf{x}} q_{\underline{w},\underline{w}^*}(\underline{x}) \leq 1$$

with equality if $\underline{w} = \underline{w}^*$ just as in (6.8) we defined for the true distribution P, and where

$$q_0(\underline{x}) = \frac{1}{\underline{w}^* \cdot \underline{x}} q(\underline{x})$$

which also sums to not more than 1 when $\underline{x}$ includes a riskless asset of return 1. Thus the wealth factor function has decomposition

$$S_T(\underline{w}) \quad = \quad e^{Ty(\underline{w}^*) - TD(q||q_{\underline{w},\underline{w}^*})}$$

$$= e^{TD(q||q_0) - TD(q||q_{\underline{w},\underline{w}^*})}$$

The second factor represents the drop in wealth from the use of a portfolio $\underline{w}$ not equal to $\underline{w}^*$.

Barron and Cover (1988) show if $\underline{w}$ is chosen to be growth optimal for a distribution P, that is, $\underline{w} = \underline{w}^*(P)$, then the drop from the maximal exponent satisfies $D(q||q_{\underline{w},\underline{w}^*}) \leq D(q||p)$. As we have shown in Lemma 6.

7.3 Portfolios of Options

Suppose we consider a single stock in the market and let x denote the stock's return. We assume that there are N possible states for x, denoted as $a_1, \ldots, a_N$, given in descending order $a_1 > a_2 > \ldots > a_N > 0$. Let a_{N+1} be a positive number less than a_N. We introduce N options, one for each state, where each share of the n^{th} option is for the right to buy a share of stock at the end of the period at a price of a_{n+1} relative to the current stock price, for $n \in \{1, \ldots, N\}$. When the stock state is x, let z_n be the return for option n. Rationally, investors do not exercise the call option if the price is lower than the strike price. Thus the return is $z_n = (x - a_{n+1})^+/v_n$, where v_n is the ratio of current option price per share to the current stock price. The positive part $(x - a_{n+1})^+$ is used to denote that the option return is zero when $x < a_{n+1}$. The vector of option returns is $\underline{z} = (z_1, \ldots, z_N)$.

Let π_n with $\sum_{n=1}^N \pi_n = 1$ denote the fraction of money to invest in option n, then $\underline{\pi} = (\pi_1, \ldots, \pi_N)$ is a portfolio on the N options. It is possible for π_n to be negative, which means that option n is shorted. Though we shall arrange that the option portfolio return $\underline{\pi} \cdot \underline{z}$ is nonnegative for all possible $\underline{z}$ (i.e., for all possible x). We also assume there is a riskless asset with constant return 1. Under a no arbitrage condition, there is no need to explicitly hold wealth in a riskless asset or the underlying stock anymore since they can be replicated by the N options. That is, there exist two portfolios $\underline{\pi}^{riskless} \in R^N$ and $\underline{\pi}^{stock} \in R^N$ on options such that for all states of the stock $\underline{\pi}^{riskless} \cdot \underline{z} = 1$ and $\underline{\pi}^{stock} \cdot \underline{z} = x$.

Hence, any linear combination of $1, x$, and coordinates of $\underline{z}$ with coefficients summing to 1 can be realized by a linear combination on $\underline{z}$ alone.

7.3.1 Compounded Wealth for Portfolios of Options

Note first that the wealth available in rebalancing between a single stock and cash (with return 1) is

$$S_T^{stock}(w) = \prod_{t=1}^{T}(1 - w + wx_t)$$

The return of the stock each period takes values in the set $\{a_1, a_2, \ldots, a_N\}$. It is $x = a_n$ when state n occurs, where $n \in \{1, \ldots, N\}$. For convenience in relating the option story to the gambling situation, we now denote the relative frequencies of occurrences of state n as $q(n)$ (rather than $q(x)$). From Theorem 6, the maximum compounded wealth in the stock and cash case (where the maximum is over all w with possible portfolio returns $(1 - w + wa_n)$ assumed to be non-negative)

$$S_T^{stock,max} = \max_w S_T^{stock}(w) = e^{Ty^*}.$$

Here, the maximum wealth portfolio weight w^* is non-zero yielding a positive $y^* = y(w^*)$ when $E_q x \neq 1$ (that is, $\sum_{n=1}^{N} q(n)a_n \neq 1$). The maximum occurs at $w^* = w^*(q)$ satisfying the properties that $q_0(w) = q(w)/(1 - w^* + w^* a_n)$ and $a_n q_0(n)$ both sum to 1. This $S_T^{stock,max}$ has a role in our wealth characterization in the case of stock options.

As we mentioned before, after the introduction of the N options, we only need to choose a portfolio $\underline{\tau}$ among these options. Importantly, there is a correspondence between the option price ratios u_n, for $n = 1, \ldots, N$ and the odds (c_n for 1) on state securities, and, moreover, for any portfolio on options, there is a corresponding $\underline{b}$ on state securities (betting fraction on "horses") such that the option return matches the gambling return, that is,

$$\underline{\pi} \cdot \underline{z} = b_n c_n \tag{7.2}$$

That there should be such a correspondence is intuitively sensible when there is a sufficiently rich collection of strike price levels for call (or put) options.

The no-arbitrage condition implies that the reciprocal odds $p^*(n) = 1/c_n$ sum to 1 ($\sum_{n=1}^{N} p^*(n) = 1$) and also that $a_n p^*(n)$ sums to 1 ($E_{p^*} x = 1$). Suppose we use portfolio $\underline{\tau}$ at periods $1, \ldots, T$ with states n_t and corresponding stock return $x_t = a_{n_t}$, and vector of option returns $\underline{z}_t$ with element $z_{n,t} = (x_t - x_{n+1})^+ / v_n$. Then, our wealth is

$$S_T(\underline{\tau}) = \prod_{t=1}^{T} \underline{\pi} \cdot \underline{z}_t.$$

Here, we allow negative π_n, provided one has the positivity of $\underline{z} \cdot \underline{\pi}$ for each possible return vector $\underline{z}$, i.e. the positivity of b_n, for $n = 1, \ldots, N$.

Theorem 7. *Under the no arbitrage condition, the compounded wealth in options is a product of three factors*

$$S_T(\underline{\tau}) = S_T^{stock,max} e^{TD(q||\hat{p^*})} e^{-TD(q||\underline{b})} \tag{7.3}$$

where $\hat{p}^*(n) = (1 - w^* + w^* a_n)p^*(n)$, *which gives* $D(q||\hat{p^*})) = 0$ *only when the odds satisfy* $p^*(n) = q(n)/(1 - w^* + w^* a_n)$. *Hence, the maximum wealth in the stock and its options is*

$$S_T^{option,max}(\underline{\tau}^*) = S_T^{stock,max} e^{TD(q||\hat{p}^*)} \tag{7.4}$$

where 1 *for* $p^*(n)$ *are the odds for state securities corresponding to option prices, for* $n = 1, \ldots, N$.

The first factor is the maximum wealth achievable investing in stock and cash only. The second factor is a higher exponential growth available precisely when the option prices are such that $(1-w^*+w^*a_n)p^*(n)$ is not equal to the relative frequencies $q(n)$, that is, when the state security reciprocal odds $p^*(n)$ are not set to be equal to $q_0(n) = q(n)/(1 - w^* + w^* a_n)$. The third factor $e^{-TD(q||\underline{b})}$ quantifies the drop in wealth by the use of an option portfolio $\underline{\pi}$ corresponding to $\underline{b}$ on state securities other than the relative frequencies q. We will show the details of proof in next section.

We can see that options provide opportunities for greater wealth than with stock and cash alone because of the positivity of the divergence $D(q||\hat{p}^*)$ when prices are set with $p^*(n)$ not equal to $q(n)/(1-w^*+w^*a_n)$. Also portfolio choice for an investor is reduced, in the case of options, to the matter of choosing betting fractions $\underline{b}$ on state securities to be close to what he believes q is likely to be.

By Lemma 6 in Chapter 6, we know that an investor who has confidence in his belief that the relative frequencies will be close to $\underline{b}$, is on one hand, encouraged to take the advantage of the options because it produces a higher growth rate by the amount $D(q||\hat{p^*})$. On the other hand, in the case of well-priced options, his drop $D(q||\underline{b})$ from the maximal exponent is greater than the drop $D(q||q_{\underline{w},\underline{w}^*})$ in the stock-cash case with $w = w^*$. Then the investor is better off with the stock-cash rebalancing alone. So if you trust that options are well-priced, you should not invest in them.

Our analysis shows that in a no-arbitrage setting, it is unwise for a broker or a firm to provide a succession of simple single period options. The reason is that fortunate investors whose portfolio corresponds to $\underline{b}$ near q would make an exponential growth of wealth off of the broker, unless the broker happens to have chosen p_n^* which turns out to match $q(n)/(1 - w^* + w^* a_n)$ associated with the relative frequencies.

Nonetheless, a broker who knows the probability beliefs of his potential investors and who believes that q will not be close to any of them is encouraged to offer options or associated gambling opportunities, because the ensuing wealth of the investor $S_T^{stock,max} e^{-TD(q||\underline{b})}$ will be less than if they had invested in the stock and cash alone lining the pockets of the broker. If regulated in a way that limits competition in offering option, a broker or firm may make money primarily off the transaction fees. The no-arbitrage requirement eliminates opportunity for option fees. In summary, options are to be played only if one has reason to believe that the other parties are less informed. They are not a financial device that should persist in an informed market.

7.3.2 Proof of Theorem 4 and Theorem 5

In this section we will prove two theorems in this chapter.

Proof of Theorem 4

$$S_T(\underline{w}) = \prod_{t=1}^{T} \underline{x}_t \cdot \underline{w} = e^{T \cdot \frac{1}{T} \sum_{t=1}^{T} \log(\underline{x}_t \cdot \underline{w})} = e^{T \sum_{\underline{x}} q(\underline{x}) \log(\underline{x} \cdot \underline{w})} = e^{T y(\underline{w})}.$$

We Notice that

$$\begin{aligned} y(\underline{w}) &= \sum_{\underline{x}} q(\underline{x}) \log(\underline{x} \cdot \underline{w}) \\ &= \sum_{\underline{x}} q(\underline{x}) \log \left(\frac{q(\underline{x})}{q(\underline{x})/(\underline{x} \cdot \underline{w}^*)} \cdot \frac{q(\underline{x})(\underline{x} \cdot \underline{w})/(\underline{x} \cdot \underline{w}^*)}{q(\underline{x})} \right) \\ &= D(q||q_0) - D(q||q_{\underline{w},\underline{w}^*}) \end{aligned}$$

where $\underline{w}^* = \underline{b}^*(q)$ satisfying

$$\sum_{\underline{x}} q(\underline{x}) x_i / (\underline{x} \cdot \underline{w}^*) = 1$$

for $i = 1, \ldots, M$, and

$$q_{\underline{w},\underline{w}^*}(\underline{x}) = \frac{\underline{w} \cdot \underline{x}}{\underline{w}^* \cdot \underline{x}} q(\underline{x})$$

and $q_0(\underline{x}) = q(\underline{x})/(\underline{w}^* \cdot \underline{x})$. Since $q_{\underline{w},\underline{w}^*}(\underline{x})$ is nonnegative and sums not more than 1, $D(q||q_{\underline{w},\underline{w}^*})$ is greater than or equal to 0 with equality if and only if $q = q_{\underline{w},\underline{w}^*}$, i. e.

$\underline{b} = \underline{b}^*$. Hence, $y(\underline{b})$ is maximized at $\underline{b} = \underline{b}^*$ and the wealth has decomposition

$$S_T(\underline{w}) = e^{TD(q||q_0)}e^{-TD(q||q_{\underline{w},\underline{w}^*})}$$

This completes the proof of Theorem 4. Let us prove Theorem 5 in the following.

Proof of Theorem 5

$$\begin{aligned}
S_T(\underline{\pi}) &= \prod_{t=1}^{T} \underline{\pi} \cdot \underline{z}_t \\
&= \prod_{t=1}^{T} b_{s_t} c_{s_t} \\
&= e^{T \cdot \frac{1}{T} \sum_{t=1}^{T} \log b_{s_t} c_{s_t}} \\
&= e^{T \sum_{n=1}^{N} q(n) \log b_n c_n} \\
&= e^{T \sum_{n=1}^{N} q(n) \log \left(\frac{q(n)}{q(n)/(1-w^*+w^* a_n)} \cdot \frac{q(n)}{(1-w^*+w^* a_n) p_n^*} \cdot \frac{b_n}{q(n)} \right)} \\
&= e^{TD(q||q_0)} e^{TD(q||\hat{p}^*)} e^{-TD(q||\underline{b})} \qquad (7.5) \\
&= S_T^{stock,max} e^{TD(q||\hat{p}^*)} e^{-TD(q||\underline{b})}
\end{aligned}$$

Since $\sum_{n=1}^{N} p_n^* = \sum_{n=1}^{N} p_n^* a_n = 1$ yields $\sum_{n=1}^{N} \hat{p}_n^* = \sum_{n=1}^{N} (1 - w^* + w^* a_n) p_n^* = 1$, then the Kullback-Leibler divergences in (7.6) are in their usual sense. Hence, the maximum wealth in options

$$\begin{aligned}
S_T(\underline{\pi}^*) &= S_T^{stock,max} e^{TD(q||\hat{p}^*)} \\
&= e^{T \sum_{n=1}^{N} q(n) \log \frac{q(n)}{q(n)/(1-b^*+b^* a_n)} + T \sum_{n=1}^{N} q(n) \log \frac{q(n)}{(1-w^*+w^* a_n) p_n^*}} \\
&= e^{T \sum_{n=1}^{N} q(n) \log \frac{q(n)}{p^*}} \\
&= e^{TD(q||p^*)}
\end{aligned}$$

where $\underline{\pi}^*$ corresponds to $\underline{w}^* = q$.

We should also notice that the odds maker do be able to set odds p_n^* for 1 such that $\hat{p}_n^* = q(n)$ and $\sum_{n=1}^N p_n^* = \sum_{n=1}^N p_n^* a_n = 1$. Hence, the minmax wealth in options equals $S_T^{stock,max}$. What's more, in this special case, we can prove the existence and uniqueness of $\underline{w}^*$ directly given the no-arbitrage condition with respect to the states which occurred during the T investment periods.

Bibliography

[1] BREIMAN, L. (1961): Optimal Gambling Systems For Favorable Games, *Fourth Berkeley Symposium on Prob. and Stat.*, 1, 65 – 78.

[2] BARRON, A.R. and T.M. COVER (1988): A Bound on the Financial Value of Information, *IEEE Trans. Info. Theory* 34(5), 1097 – 1100.

[3] BERNSTEIN, W.J. and WILKINSON, D. (1997): Diversification, Rebalancing and the Geometric Mean Frontier, *Online Journal of Efficient Frontier.*

[4] BOYD, S. and L. VANDENBERGHE (2004): *Convex Optimization*, Cambridge University Press.

[5] COVER, T.M. (1991): Univsersal Portfolios, *Math. Finance* 1(1),1 – 29.

[6] COVER, T.M. and E. ORDENTLICH (1996): Universal Portfolio wilth Side Information, *IEEE Trans. Info. Theory*, 42(2), 348 – 363.

[7] CROSS, J. and A.R. BARRON (2003): Efficient Universal Portfolios For Past-dependent Target Classes, *Math. Finance*, 13(2), 245 – 276.

[8] CHAN, L., N. JEGADEESH and J. LAKONISHOK (1996): Momentum Strategies, *Journal of Finance* 51(5), 1681 – 1713.

[9] ELTON, E.J and MARTIN, J.G. (1974), On the Optimality of Some Multi-period Selection Criteria, *Journal of Business*, 47(2), 231 – 243.

[10] FIACCO, A.V. and MCCORMICK, G.P. (1968), *Nonlinear Programming: Sequential Unconstraned Minimization Techniques*, Wiley and Sons.

[11] FISHBURN, P.C. (1970), *Utility for Decision Making*, Krieger Publishing Co.

[12] GOETZMANN, W.N. (Online Book): *An Introduction to Investment Theory*, Yale School of Management.

[13] HELMBOLD, D.P., R.E. SCHAPIRE, Y. SINGER and M.K. WARMUTH (1998): On-line Portfolio Selection Uing Multiplicative Updates, *Math. Finance*, 8(4), 325 – 347.

[14] HAKANSSON, N.H. (1971), Capital Growth and the Mean-Variance Approach to Portfolio Selection, *Journal of Financial and Quantitative Analysis*, 6(1), 517 – 558.

[15] HAKANSSON, N.H. (1979), A Characterization of Optimal Multiperiod Portfolio Policies, *TIMS Studies in Management Sciences*, 11, 169 – 177.

[16] ISHIJIMA H. (2001): Nummerical Methods for Universal Portfolios, *Quantitative Methods in Finance Conference.*

[17] JEGADEESH, N. (1990): Evidence of Predictable Behavior of security Returns, *Journal of Finance* 45(3), 881 – 898.

[18] JEDADEESH, N. and S. TITMAN (1993): Returns to Buying Winners and Selling Losers: Implications for Stock Market Efficiency, *Journal of Finance* 48(1), 65–91.

[19] KARMARKAR, N. (1984): *A New Polynomial-Time Algorithm for Linear Programming*, Combinatorica 4, 373 – 395.

[20] KREPS, D.M. (1988), *Notes on Theory of Choice*, Westview Press.

[21] KELLY, J.L. (1956), A New Interpretaion of Information Rate, *IEEE Trans. Info. Theory*, 2(3), 185 – 189.

[22] LI J. and A.R. BARRON (2000): Mixture Density Estimation, *Advances in Neural Information Processing Systems*, 12, 279 – 285, MIT Press.

[23] LEHMANN, B. (1990): Fads, Martingales, and Market Efficiency, *Quarterly Journal of Economics* 60(1), 1 – 28.

[24] LUENBERGER, D.G. (1997): *Investment Science*, Oxford University Press.

[25] LATANE, H.A. (1959), Criteria for Choice Among Risky Ventures, *Journal of Political Economy*, 67(2), 144 – 155.

[26] LATANE, H.A. and TUTTLE, D.L. (1967), Criteria for Portfolio Building, *Journal of Finance*, 22(3), 359 – 373

[27] MARKOWITZ, H.M. (1952): Portfolio Selection, *Journal of Finance*, 7(1), 77–91.

[28] MACBETH, J.D. (1995), What's the Long-term Expected Return to Your Portfolio?, *Financial Analyst Journal*, 51(5), 6 – 8.

[29] NESTROV, Y. and A. NEMIROVSKY (1993): Interior-Point Polinomial Algorithms in Convesx Programming, *SIAM Studies in Applied Mathematics.*

[30] NEUMANN, J.V. and MORGENSTERN, O. (1944), *Theory of Games and Economic Behavior*, Princeton University Press.

[31] NEUMANN, J.V. and MORGENSTERN, O. (1947), *Theory of Games and Economic Behavior 2nd edition*, Princeton University Press.

[32] SHARPE, W.F. (1985): *Investment*, Prentice-Hall, Englewood Cliffs.

[33] XIE, Q. and A.R. BARRON (2000): Asymptotic Minimax Regret for Data Compression, Gambling and Prediction, *IEEE Trans. Info. Theory*, 46(2) 431 – 445.

[34] YUDIN, D. and A. NEMIROVSKY (1976): Evaluation of the Information Complexity of Mathematical Programming Problems, (in Russian), *Ekonomikai Matematicheskie Metody* 12, 128 – 142, (English translation: *Matekon* 13(2), 3 – 45).

[35] YUDIN, D. and A. NEMIROVSKY (1983): *Problem Complxity and Method Efficiency in Optimization*, New York: Willey.

[36] YU, J. (2003), *Maximal Compounded Wealth for Portfolios of Stocks and Options*, Ph.D. Dissertation Prospectus, Department of Statistics, Yale University.