Towards Optimal Estimation of Bivariate Isotonic Matrices with Unknown Permutations

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Abstract

Many applications, including rank aggregation, crowd-labeling, and graphon estimation, can be modeled in terms of a bivariate isotonic matrix with unknown permutations acting on its rows and columns. We consider the problem of estimating such a matrix based on noisy observations of a subset of its entries, and design and analyze polynomial-time algorithms that improve upon the state of the art. In particular, our results imply that any such $n \times n$ matrix can be estimated efficiently in the normalized, squared Frobenius norm at rate $\tilde{O}(n^{-3/4})$, thus narrowing the gap between $\tilde{O}(n^{-1})$ and $\tilde{O}(n^{-1/2})$, hitherto the rates of the most statistically and computationally efficient methods, respectively. Additionally, our algorithms are minimax optimal in another natural metric that measures how well an estimate captures each row of the matrix. Along the way, we prove matching upper and lower bounds on the minimax radii of certain cone testing problems, which may be of independent interest.

1 Introduction

Structured matrices with unknown permutations acting on their rows and columns arise in multiple applications, including estimation from pairwise comparisons [BT52, SBGW17] and crowd-labeling [DS79, SBW16b]. Traditional parametric models (e.g., [BT52, Luc59, Thu27, DS79]) assume that these matrices are obtained from rank-one matrices via a known link function. Aided by tools such as maximum likelihood estimation and spectral methods, researchers have made significant progress in studying both statistical and computational aspects of these parametric models [HOX14, RA14, SBB+16, NOS16, ZCZJ16, GLZ16, KOS11b, LPI12, DDKR13, GKM11] and their low-rank generalizations [RA16, NOTX17, KOS11a].

On the other hand, evidence from empirical studies suggests that real-world data is not always well-described by such parametric models [ML65, BW97]. With the goal of increasing model flexibility, a recent line of work has studied the class of permutation-based models [Cha15, SBGW17, SBW16b]. Rather than imposing parametric conditions on the matrix entries, these models impose only shape constraints on the matrix, such as monotonicity, before unknown permutations act on the rows and columns of the matrix. On one hand, this more flexible class reduces modeling bias compared to its parametric counterparts while, perhaps surprisingly, producing models that can be estimated at rates that differ only by logarithmic factors from the classical parametric models. On the other hand, these advantages of permutation-based models are accompanied by significant computational challenges. The unknown permutations make the parameter space highly non-convex,
so that efficient maximum likelihood estimation is unlikely. Moreover, spectral methods are often suboptimal in approximating shape-constrained sets of matrices [Cha15, SBGW17]. Consequently, results from many recent papers show a non-trivial statistical-computational gap in estimation rates for models with latent permutations [SBGW17, CM16, SBW16b, FMR16, PWC17].

**Related work.** While the main motivation of our work comes from nonparametric methods for aggregating pairwise comparisons, we begin by discussing a few other lines of related work. The current paper lies at the intersection of shape-constrained estimation and latent permutation learning. Shape-constrained estimation has long been a major topic in nonparametric statistics, and of particular relevance to our work is the estimation of a bivariate isotonic matrix without latent permutations [CGS18]. There, it was shown that the minimax rate of estimating an $n \times n$ matrix from noisy observations of all its entries is $\tilde{\Theta}(n^{-1})$. The upper bound is achieved by the least squares estimator, which is efficiently computable due to the convexity of the parameter space.

Shape-constrained matrices with permuted rows or columns also arise in applications such as seriation [FJBd13, FMR16], feature matching [CD16], and graphon estimation [BCL11, CA14, GLZ15, KTV17]. In particular, the monotone subclass of the statistical seriation model [FMR16] contains $n \times n$ matrices that have increasing columns, and an unknown row permutation. The authors established the minimax rate $\tilde{\Theta}(n^{-2/3})$ for estimating matrices in this class and proposed a computationally efficient algorithm with rate $\tilde{O}(n^{-1/2})$. For the subclass of such matrices where in addition, the rows are also monotone, the results of the current paper improve the two rates to $\tilde{\Theta}(n^{-1})$ and $\tilde{O}(n^{-3/4})$ respectively.

Graphon estimation has seen its own extensive literature, and we only list those papers that are most relevant to our setting. In essence, these problems involve non-parametric estimation of a bivariate function $f$ from noisy observations of $f(\xi_i, \xi_j)$ with the design points $\{\xi_i\}_{i=1}^n$ drawn i.i.d. from some distribution supported on the interval $[0,1]$. In contrast to non-parametric estimation, however, the design points in graphon estimation are unobserved, which gives rise to the underlying latent permutation. Modeling the function $f$ as monotone recovers the model studied in this paper, but other settings have been studied by various authors: notably those where the function $f$ is Lipschitz [CA14], block-wise constant [BCL11, GLZ15, KTV17] (also known as the stochastic block model [Abb17]), or with $f$ satisfying other smoothness assumptions [WO13, GLZ15, BCCG15].

Another related model in the pairwise comparison literature is that of noisy sorting [BM08], which involves a latent permutation but no shape-constraint. In this prototype of a permutation-based ranking model, we have an unknown, $n \times n$ matrix with constant upper and lower triangular portions whose rows and columns are acted upon by an unknown permutation. The hardness of recovering any such matrix in noise lies in estimating the unknown permutation. As it turns out, this class of matrices can be estimated efficiently at minimax optimal rate $\tilde{\Theta}(n^{-1})$ by multiple procedures: the original work by Braverman and Mossel [BM08] proposed an algorithm with time complexity $O(n^c)$ for some unknown and large constant $c$, and recently, an $\tilde{O}(n^2)$-time algorithm was proposed by Mao et al. [MWR17]. These algorithms, however, do not generalize beyond the noisy sorting class, which constitutes a small subclass of an interesting class of matrices that we describe next.

The most relevant body of work to the current paper is that on estimating matrices satisfying the **strong stochastic transitivity** condition, or SST for short. This class of matrices contains all $n \times n$ bivariate isotonic matrices with unknown permutations acting on their rows and columns, with an additional skew-symmetry constraint. The first theoretical study of these matrices was carried
out by Chatterjee [Cha15], who showed that a spectral algorithm achieved the rate $\tilde{O}(n^{-1/4})$ in the normalized, squared Frobenius norm. Shah et al. [SBGW17] then showed that the minimax rate of estimation is given by $\tilde{\Theta}(n^{-1})$, and also improved the analysis of the spectral estimator of Chatterjee to obtain the computationally efficient rate $\tilde{O}(n^{-1/2})$. In follow-up work [SBW16a], they also showed a second CRL estimator based on the Borda count that achieved the same rate, but in near-linear time. In related work, Chatterjee and Mukherjee [CM16] analyzed a variant of the CRL estimator, showing that for subclasses of SST matrices, it achieved rates that were faster than $O(n^{-1/2})$. In a complementary direction, a superset of the current authors [PMM++17] analyzed the estimation problem under an observation model with structured missing data, and showed that for many observation patterns, a variant of the CRL estimator was minimax optimal.

Shah et al. [SBW16a] also showed that conditioned on the planted clique conjecture, it is impossible to improve upon a certain notion of adaptivity of the CRL estimator in polynomial time. Such results have prompted various authors [FMR16, SBW16a] to conjecture that a similar statistical-computational gap also exists when estimating SST matrices in the Frobenius norm.

**Our contributions.** The main contribution of the current paper is to tighten the aforementioned statistical-computational gap. More precisely, we study the problem of estimating a bivariate isotonic matrix with unknown permutations acting on its rows and columns, given noisy, partial observations of its entries; this matrix class strictly contains the SST model [Cha15, SBGW17] for ranking from pairwise comparisons. As a corollary of our results, we show that when the underlying matrix has dimension $n \times n$ and $\Theta(n^2)$ noisy entries are observed, our polynomial-time, two-dimensional sorting algorithm provably achieves the rate of estimation $\tilde{O}(n^{-3/4})$ in the normalized Frobenius norm; thus, this result breaks the previously mentioned $\tilde{O}(n^{-1/2})$ barrier [SBGW17, CM16]. Although the rate $\tilde{O}(n^{-3/4})$ still differs from the minimax optimal rate $\tilde{\Theta}(n^{-1})$, our algorithm is, to the best of our knowledge, the first efficient procedure to obtain a rate faster than $\tilde{O}(n^{-1/2})$ uniformly over the SST class. This guarantee, which is stated in slightly more technical terms below, can be significant in practice (see Figure 1).

**Main theorem (informal).** There is an estimator $\hat{M}$ computable in time $O(n^{2.5})$ such that for any $n \times n$ SST matrix $M^\star$, given $\Theta(n^2)$ Bernoulli observations of its entries, we have

$$E \left[ \frac{1}{n^2} \| \hat{M} - M^\star \|_F^2 \right] \leq C \left( \frac{\log n}{n} \right)^{3/4}.$$

Additionally, we also analyze the case where the columns (or rows) of the underlying matrix are perfectly ordered, and introduce a new error metric that quantifies how well each row (or column) of the matrix can be estimated. We show that our algorithms are minimax optimal (up to logarithmic factors) in this metric for both matrix classes of interest, and our bounds shed light on why a statistical-computational gap was conjectured in the first place.

Our two algorithms are novel in the sense that they are neither spectral in nature, nor simple variations of the Borda count estimator that was previously employed. Our algorithms take advantage of the fine monotonicity structure of the underlying matrix along both dimensions, and this allows us to prove tighter bounds than before. In addition to making algorithmic contributions, we also briefly revisit the minimax rates of estimation.
Figure 1: **Left:** A bivariate isotonic matrix; the ground truth $M^* \in [0,1]^{n \times n}$ is a row and column permuted version of such a matrix. **Right:** A log-log plot of the error $\frac{1}{n^2} \| \hat{M} - M^* \|^2_{F}$ (averaged over 10 experiments each using $n^2$ Bernoulli observations) of our estimator and the CRL estimator [SBW16a].

**Organization.** In Section 2, we formally introduce our estimation problem, and describe in detail how it is connected to applications in crowd-labeling and ranking from pairwise comparisons. Section 3 contains precise statements and discussions of our main results, and we provide proofs of our main results in Section 4.

**Notation.** For a positive integer $n$, let $[n] := \{1, 2, \ldots, n\}$. For a finite set $S$, we use $|S|$ to denote its cardinality. For two sequences $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$, we write $a_n \lesssim b_n$ if there is a universal constant $C$ such that $a_n \leq C b_n$ for all $n \geq 1$. The relation $a_n \gtrsim b_n$ is defined analogously. We use $c, C, c_1, c_2, \ldots$ to denote universal constants that may change from line to line. We use $\text{Ber}(p)$ to denote the Bernoulli distribution with success probability $p$, the notation $\text{Bin}(n,p)$ to denote the binomial distribution with $n$ trials and success probability $p$, and the notation $\text{Poi}(\lambda)$ to denote the Poisson distribution with mean $\lambda > 0$. Given a matrix $M \in \mathbb{R}^{n_1 \times n_2}$, its $i$-th row is denoted by $M_i$. Let $\mathcal{S}_n$ denote the set of all permutations $\pi : [n] \to [n]$. Let $\text{id}$ denote the identity permutation, where the dimension can be inferred from context.

## 2 Background and problem setup

In this section, we present the relevant technical background and notation on permutation-based models, and introduce the observation model and error metrics of interest. We also elaborate on how exactly these models arise in practice.

### 2.1 Matrix models

Our main focus is on designing efficient algorithms for estimating a bivariate isotonic matrix with unknown permutations acting on its rows and columns. Formally, we define $\mathbb{C}_{BISO}$ to be the class
of matrices in $[0,1]^{n_1 \times n_2}$ with nondecreasing rows and nondecreasing columns. For readability, we assume throughout that $n_1 \geq n_2$ unless otherwise stated; our results can be straightforwardly extended to the other case. Given a matrix $M \in \mathbb{R}^{n_1 \times n_2}$ and permutations $\pi \in \mathfrak{S}_{n_1}$ and $\sigma \in \mathfrak{S}_{n_2}$, we define the matrix $M(\pi, \sigma) \in \mathbb{R}^{n_1 \times n_2}$ by specifying its entries as

$$[M(\pi, \sigma)]_{i,j} = M_{\pi(i),\sigma(j)} \text{ for } i \in [n_1], j \in [n_2].$$

Also define the class $C_{\text{BISO}}(\pi, \sigma) := \{M(\pi, \sigma) : M \in C_{\text{BISO}}\}$ as the set of matrices that are bivariate isotonic when viewed along the row permutation $\pi$ and column permutation $\sigma$, respectively.

The classes of matrices that we are interested in estimating are given by

$$C^r_{\text{Perm}} := \bigcup_{\pi \in \mathfrak{S}_{n_1}, \sigma \in \mathfrak{S}_{n_2}} C_{\text{BISO}}(\pi, \sigma), \text{ and its subclass } C^r_{\text{Perm}} := \bigcup_{\pi \in \mathfrak{S}_{n_1}} C_{\text{BISO}}(\pi, \text{id}).$$

The former class contains bivariate isotonic matrices with both rows and columns permuted, and the latter contains those with only rows permuted.

### 2.2 Observation models

In order to study estimation from noisy observations of a matrix $M^*$ in either of the classes $C^r_{\text{Perm}}$ or $C^r_{\text{Perm}}$, we suppose that $N$ noisy entries are sampled independently and uniformly with replacement from all entries of $M^*$. This sampling model is popular in the matrix completion literature, and is a special case of the trace regression model [NW12, KLT11]. It has also been used in the context of permutation-based models by Mao et al. [MWR17] to study the noisy sorting class.

More precisely, let $E^{(i,j)}$ denote the $n_1 \times n_2$ matrix with 1 in the $(i,j)$-th entry and 0 elsewhere, and suppose that $X_\ell$ is a random matrix sampled independently and uniformly from the set $\{E^{(i,j)} : i \in [n_1], j \in [n_2]\}$. We observe $N \leq n_1 n_2$ independent pairs $\{(X_\ell, y_\ell)\}_{\ell=1}^N$ from the model

$$y_\ell = \text{tr}(X_\ell^\top M^*) + z_\ell, \quad (1)$$

where the observations are contaminated by independent, centered, sub-Gaussian noise $z_\ell$ with variance parameter $\zeta^2$. Besides the standard Gaussian observation model, in which

$$z_\ell \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1), \quad (2a)$$

another noise model of interest is one which arises in applications such as crowd-labeling and ranking from pairwise comparisons. Here, our observations take the form

$$y_\ell \sim \text{Ber}(\text{tr}(X_\ell^\top M)), \quad (2b)$$

and consequently, the sub-Gaussian parameter $\zeta^2$ is bounded. For a discussion of other regimes of noise in a related matrix model, see Gao [Gao17].

For analytical convenience, we employ the standard trick of Poissonization, whereby we assume throughout the paper that $N' = \text{Poi}(N)$ random observations are drawn according to the trace regression model (1). Upper and lower bounds derived under this model carry over with loss of constant factors to the model with exactly $N$ observations; for a detailed discussion, see Appendix B.

For notational convenience, denote the probability that an entry of the matrix is observed under Poissonized sampling by $p_{\text{obs}} = 1 - \exp(-N/n_1 n_2)$. Since we assume throughout that $N \leq n_1 n_2$, it can be verified that $N/n_1 n_2 \leq p_{\text{obs}} \leq N/n_1 n_2$. 

Now given $N' = \text{Poi}(N)$ observations $\{(X_\ell, y_\ell)\}_{\ell=1}^{N'}$, let us define the matrix of observations $Y = Y \left(\{(X_\ell, y_\ell)\}_{\ell=1}^{N'}\right)$, with entry $(i, j)$ given by

$$Y_{i,j} = \frac{1}{p_{\text{obs}}} \frac{1}{\sum_{\ell=1}^{N'} 1\{X_\ell = E(i,j)\}} \sum_{\ell=1}^{N'} y_\ell 1\{X_\ell = E(i,j)\}. \quad (3)$$

In words, the rescaled entry $p_{\text{obs}} Y_{i,j}$ is the average of all the noisy realizations of $M_{i,j}^*$ that we have observed, or zero if the entry goes unobserved. Note that $E[Y_{i,j}] = \frac{1}{p_{\text{obs}}} M_{i,j}^* \cdot p_{\text{obs}} = M_{i,j}^*$, so that $E[Y] = M^*$. Moreover, we may write the model in the linearized form $Y = M^* + W$, where $W$ is a matrix of additive noise having independent, zero-mean, sub-Gaussian entries.

### 2.3 Error metrics

We analyze estimation of the matrix $M^*$ and the permutations $(\pi^*, \sigma^*)$ in two metrics. For a tuple of proper estimates $(\hat{M}, \hat{\pi}, \hat{\sigma})$, where $\hat{M} = \hat{M}(\hat{\pi}, \hat{\sigma}) \in \mathbb{C}_{\text{BISO}}(\hat{\pi}, \hat{\sigma})$ (and $\hat{\sigma} = \text{id}$ if we are estimating over the class $\mathbb{C}_{\text{perm}}^r$), the normalized squared Frobenius error is given by the random variable

$$\mathcal{F}(M^*, \hat{M}) = \frac{1}{n_1 n_2} \| M^* - \hat{M} \|_F^2.$$

The row-wise approximation error of the estimate $\hat{\pi}$, on the other hand, is given by the random variable

$$\mathcal{R}(M^*, \hat{\pi}) = \max_{i \in [n_1]} \mathcal{R}_i(M^*, \hat{\pi}), \text{ where }$$

$$\mathcal{R}_i(M^*, \hat{\pi}) = \frac{1}{n_2} \| M^*(\pi^*, \sigma^*) \}_i - [M^*(\hat{\pi}, \hat{\sigma}) \}_i \|_2^2.$$

As will be clear from the sequel, the quantity $\mathcal{R}_i(M^*, \hat{\pi})$ arises as a natural consequence of our development; it represents the approximation error of the permutation estimate $\hat{\pi}$ on row $i$ of the matrix $M^*(\pi^*, \sigma^*)$.

When estimating over the class $\mathbb{C}_{\text{perm}}^r$, the column-wise approximation error $\mathcal{C}(M^*, \hat{\sigma})$ is defined analogously as $\mathcal{C}(M^*, \hat{\sigma}) = \max_{i \in [n_2]} \mathcal{C}_i(M^*, \hat{\sigma})$, where $\mathcal{C}_i(M^*, \hat{\sigma}) = \frac{1}{n_1} \| M^*(\pi^*, \sigma^*) \}_i - [M^*(\hat{\pi}, \hat{\sigma}) \}_i \|_2^2$, and we have used $M^i$ to denote the $i$th column of a matrix $M$. However, since the error $\mathcal{C}$ can be shown to exhibit similar behavior to the error $\mathcal{R}$, it suffices to study the row-wise error $\mathcal{R}$ defined above. The relation between the error metrics for a natural class of algorithms is shown in more rigorous terms by Proposition 1.

### 2.4 Applications

We now discuss in detail how the matrix models studied in this paper arise in crowd-labeling and estimation from pairwise comparisons. The class $\mathbb{C}_{\text{perm}}^r$ was studied as a permutation-based model for crowd-labeling [SBW16b] in the case of binary questions, and was proposed as a strict generalization of the classical Dawid-Skene model [DS79, KOS11b, LPI12, DDKR13, GKM11]. Here there is a set of $n_2$ questions of a binary nature; the true answers to these questions can be represented by a vector $x^* \in \{0, 1\}^{n_2}$, and our goal is to estimate this vector by asking these questions to $n_1$ workers on a crowdsourcing platform. A key to this problem is being able to model
the probabilities with which workers answer questions correctly, and we do so by collecting these
probabilities within a matrix $M^* \in [0,1]^{n_1 \times n_2}$. Assuming that workers have a strict ordering $\pi$
of their abilities, and that questions have a strict ordering $\sigma$ of their difficulties, the matrix $M^*$
is bivariate isotonic when the rows are ordered in increasing order of worker ability, and columns
are ordered in decreasing order of question difficulty. However, since worker abilities and question
difficulties are unknown a priori, the matrix of probabilities obeys the inclusion $M^* \in C_{r,c}^{\pi,\sigma}$.

In the calibration problem, we would like to ask questions whose answers we know a priori, so
that we can estimate worker abilities and question difficulties, or more generally, the entries of the
matrix $M^*$. This corresponds to estimating matrices in the class $C_{r,c}^{\pi,\sigma}$ from noisy observations of
their entries. In the particular case where in addition to the true answers, we also know the relative
difficulties of the questions themselves, we may assume that the column permutation is known, so
that our estimation problem is now over the class $C_{\pi,\sigma}^\pi$.

A subclass of $C_{\pi,\sigma}^{\pi,\sigma}$ specializes to the case $n_1 = n_2 = n$, and also imposes an additional skew
symmetry constraint. More precisely, define $C_{\pi,\pi}^{\pi,\pi}$ analogously to the class $C_{\pi,\pi}^\pi$, except with
matrices having columns that are nonincreasing instead of nondecreasing. Also define the class
$C_{\pi,\pi}^{\pi,\pi}(n) := \{M \in [0,1]^{n \times n} : M + M^T = 11^T\}$, and the strong stochastic transitivity class
$C_{\pi,\pi}^{\pi,\pi}(n) := \left( \bigcup_{\pi \in S_n} C_{\pi,\pi}^{\pi,\pi}(\pi,\pi) \right) \cap C_{\pi,\pi}^{\pi,\pi}(n)$.

The class $C_{\pi,\pi}^{\pi,\pi}(n)$ is useful as a model for estimation from pairwise comparisons [Cha15,
SBGW17], and was proposed as a strict generalization of parametric models for this problem [BT52,
NOS16, RA14]. In particular, given $n$ items obeying some unknown underlying ranking $\pi$, entry
$(i,j)$ of a matrix $M^* \in C_{\pi,\pi}^{\pi,\pi}(n)$ represents the probability $\Pr(i \succ j)$ with which item $i$ beats item $j$
in a comparison. The shape constraint encodes the transitivity condition that for all triples $(i,j,k)$
obeying $\pi(i) < \pi(j) < \pi(k)$, we must have
$$\Pr(i \succ k) \geq \max\{\Pr(i \succ j), \Pr(j \succ k)\}.$$ 

For a more classical introduction to these models, see the papers [Fis73, ML65, BW97] and the
references therein. Our task is to estimate the underlying ranking from results of passively chosen
pairwise comparisons\(^1\) between the $n$ items, or more generally, to estimate the underlying probabilities $M^*$
that govern these comparisons\(^2\). In particular, the underlying probabilities could be estimated globally, as reflected in the Frobenius error $F$, or locally, as reflected in the maximal
row-wise error $R$. In the latter case, we require that for each $k$, an estimate of the $k$-th ranked item
must be “close” to the $k$-th ranked item in ground truth. Here, items $i$ and $j$ are said to be
close if the vector of probabilities with which item $i$ beats other items is similar to the vector of
probabilities with which item $j$ beats other items.

All results in this paper stated for the more general matrix model $C_{\pi,\pi}^{\pi,\pi}$ apply to the class
$C_{\pi,\pi}^{\pi,\pi}(n)$ with minimal modifications.

\(^1\)Such a passive, simultaneous setting should be contrasted with the active case (e.g., [HSRW16, FOPS17,
AAAK17]), where we may sequentially choose pairs of items to compare depending on the results of previous
comparisons.

\(^2\)Accurate, proper estimates of $M^*$ in the Frobenius error metric translate to accurate estimates of the ranking $\pi$
(see Shah et al. [SBGW17]).
3 Main results

In this section, we present precise statements of our main results. We assume throughout this section that as per the setup, we have \( n_1 \geq n_2 \) and \( N \in [n_1n_2] \). A summary of our results is provided in Table 1 for the special case of square, \( n \times n \) matrices, with \( N = \Theta(n^2) \) observations. We begin by briefly revisiting the fundamental limits of estimation in the Frobenius error, and proving minimax lower bounds in the row-wise approximation error. We then introduce our algorithms in Section 3.2.

<table>
<thead>
<tr>
<th>Model Class ( \rightarrow ) Metric ↓</th>
<th>Class ( \mathbb{C}^r_{\text{Perm}} )</th>
<th>Class ( \mathbb{C}^{c,c}_{\text{Perm}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frobenius estimation error ( \mathbb{E} [F] )</td>
<td>( \Theta(n^{-1}) ) Theorem 1</td>
<td>( \Theta(n^{-1}) ) Theorem 1; [SBGW17]</td>
</tr>
<tr>
<td>Row-wise approximation error ( \mathbb{E} [R] )</td>
<td>( \Theta(n^{-3/4}) ) Theorem 2</td>
<td>( \Theta(n^{-1}/2) ) Theorem 4; [SBW16a, CM16]</td>
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Table 1: Estimation rates for each model class and metric for \( n \times n \) matrices with Bernoulli or Gaussian observations of \( n^2 \) entries. (Theorem numbers reference the present paper.)

### 3.1 Statistical limits of estimation

We begin by characterizing the fundamental limits of estimation under the trace regression observation model (1) with \( N' = \text{Poi}(N) \) observations. We define the least squares estimator over a closed set \( \mathbb{C} \) of \( n_1 \times n_2 \) matrices as the projection

\[
\hat{M}_{\text{LS}}(\mathbb{C}, Y) \in \arg \min_{M \in \mathbb{C}} \| Y - M \|_F^2.
\]

The projection is a non-convex problem when the class \( \mathbb{C} \) is given by either the class \( \mathbb{C}^r_{\text{Perm}} \) or \( \mathbb{C}^{c,c}_{\text{Perm}} \), and is unlikely to be computable exactly in polynomial time. However, studying this estimator allows us to establish a baseline that characterizes the best achievable statistical rate. The following theorem characterizes its Frobenius risk, showing that the least squares estimator minimax optimal up to a logarithmic factor in the dimension; recall the shorthand \( Y = Y \left( \{ X_\ell, y_\ell \}_{\ell=1}^{N'} \right) \).

**Theorem 1.** For any matrix \( M^* \in \mathbb{C}^{r,c}_{\text{Perm}} \), we have

\[
\mathbb{F}(M^*, \hat{M}_{\text{LS}}(\mathbb{C}^{c,c}_{\text{Perm}}, Y)) \lesssim (\zeta^2 \lor 1) \frac{n_1 \log^2 n_1}{N} \tag{4a}
\]

with probability at least \( 1 - (n_1n_2)^{-3} \).

Additionally, under the standard Gaussian observation model (2a) or the Bernoulli observation model (2b), and provided \( N \gtrsim n_1 \), any estimator \( \hat{M} \) satisfies

\[
\sup_{M^* \in \mathbb{C}_{\text{Perm}}} \mathbb{E} \left[ \mathbb{F}(M^*, \hat{M}) \right] \gtrsim \frac{n_1}{N}. \tag{4b}
\]
The factor \((\zeta^2 \vee 1)\) appears in the upper bound instead of the noise variance \(\zeta^2\) because even if the noise is zero, there are missing entries. It is also worth mentioning that the assumption that \(N \gtrsim n_1\) in the minimax lower bound is not restrictive, since the bounds (4a) and (4b) are vacuous otherwise, with the risk of any estimator lower bounded by a constant. Via the inclusion \(C_{\text{Perm}} \subseteq C_{\text{Permc}}\), the upper and lower bounds given by equations (4a) and (4b) extend to the classes \(C_{\text{Perm}}\) and \(C_{\text{Permc}}\), respectively, and we have thus characterized the minimax rate of estimation in the Frobenius norm for both classes up to a logarithmic factor.

We now turn to establishing minimax lower bounds for estimating matrices in both classes in the row-wise approximation metric.

**Theorem 2.** Suppose that \(N \gtrsim n_1\) samples are drawn from either the standard Gaussian observation model (2a) or the Bernoulli observation model (2b). Then:

(a) When estimating over the class \(C_{\text{Permc}}\), any permutation estimate \(\hat{\pi}\) incurs worst case error

\[
\sup_{M^* \in C_{\text{Permc}}} \mathbb{E}[R(M^*, \hat{\pi})] \gtrsim \left(\frac{n_1}{N}\right)^{1/2}.
\]  

(b) When estimating over the class \(C_{\text{Perm}}\), any permutation estimate \(\hat{\pi}\) incurs worst case error

\[
\sup_{M^* \in C_{\text{Perm}}} \mathbb{E}[R(M^*, \hat{\pi})] \gtrsim \left(\frac{n_1}{N}\right)^{3/4}.
\]

As before, the assumed lower bound on the sample size \(N\) is not restrictive. The bounds (5a) and (5b) are optimal up to logarithmic factors and attained by estimators that are computable in polynomial time: we establish these facts in Theorems 3 and 4 to follow. Thus, leveraging perfect knowledge of the column permutation can significantly improve estimates of the row permutation, even in the row-wise estimation metric \(R\).

The lower bounds of Theorem 2 are proved via reductions to particular hypothesis testing problems, for which we prove lower bounds on the minimax testing radii when the entries are partially observed. In particular, in proving the bound (5b), we also lower bound the minimax radius of testing, given noisy observations of two vectors in the positive, monotone cone, whether or not one of the vectors is entry-wise larger than the other. Bound (5a) follows from an extension, to the partially observed setting, of existing lower bounds on the minimax radius of testing the positive orthant cone against the zero vector [WWG17].

It is also important to mention that some algorithms in the literature [SBW16a, CM16] are already able to match the lower bound (5a) up to a logarithmic factor, and their analysis was recently improved to remove the logarithmic factor by a superset of the current authors [PMM+17].

### 3.2 Efficient algorithms

Our algorithms belong to a broader family of algorithms that rely on two distinct steps: first, estimate the unknown permutation(s) defining the problem; then project onto the class of matrices that are bivariate isotonic when viewed along the estimated permutations. Formally, any such algorithm is described by the meta-algorithm below.

**Algorithm 1 (meta-algorithm)**
• Step 0: Split the observations into two disjoint parts, each containing \( N' \) observations, and construct the matrices \( Y^{(1)} = Y\left( \{X_\ell, y_\ell\}_{\ell=1}^{N'/2} \right) \) and \( Y^{(2)} = Y\left( \{X_\ell, y_\ell\}_{\ell=N'/2+1}^{N'} \right) \).

• Step 1: Use \( Y^{(1)} \) to obtain the estimates \((\hat{\pi}, \hat{\sigma})\), setting \( \hat{\sigma} = \text{id} \) if estimating over class \( \mathcal{C}_{\text{Perm}}^{r,c} \). Return the matrix estimate \( \hat{M}(\pi, \sigma) := \arg \min_{M \in \mathcal{C}_{\text{BIso}}(\hat{\pi}, \hat{\sigma})} \|Y^{(2)} - M\|_F^2 \).

Owing to the convexity of the set \( \mathcal{C}_{\text{BIso}}(\pi, \sigma) \), the projection operation in Step 2 of the algorithm can be computed in near linear time [BDPR84, KRS15]. The following result, a variant of Proposition 4.2 of Chatterjee and Mukherjee [CM16], allows us to characterize the error rate of any such meta-algorithm by a function of the permutation estimates \((\hat{\pi}, \hat{\sigma})\).

Recall the definition of the set \( \mathcal{C}_{\text{BIso}}(\pi, \sigma) := \{M(\pi, \sigma) : M \in \mathcal{C}_{\text{BIso}}\} \) as the set of matrices that are bivariate isotonic when viewed along the row permutation \( n \) and column permutation \( \sigma \), respectively.

**Proposition 1.** Suppose that \( M^* \in \mathcal{C}_{\text{BIso}}(\pi^*, \sigma^*) \) where \( \pi^* \) and \( \sigma^* \) are unknown permutations in \( \mathcal{S}_{n_1} \) and \( \mathcal{S}_{n_2} \), respectively. Then with probability at least \( 1 - (n_1 n_2)^{-3} \), the estimate \( \hat{M}(\hat{\pi}, \hat{\sigma}) \) obtained by running the meta-algorithm satisfies

\[
\mathcal{F}(M^*, \hat{M}(\hat{\pi}, \hat{\sigma})) \lesssim (\zeta^2 \vee 1) \left( \frac{n_1 \log^2 n_1}{N} \right) + \frac{1}{n_1 n_2} \|M^*(\pi^*, \sigma^*) - M^*(\hat{\pi}, \hat{\sigma})\|_F^2 + \frac{1}{n_1 n_2} \|M^*(\pi^*, \sigma^*) - M^*(\pi^*, \hat{\sigma})\|_F^2.
\]

(6)

A few comments are in order. The first term on the right hand side of the bound (6) corresponds to an estimation error, if the true permutations \( \pi \) and \( \sigma \) were known a priori, and the latter two terms correspond to an approximation error that we incur as a result of having to estimate these permutations from data. Comparing the bound (6) to the minimax lower bound (4b), we see that up to a logarithmic factor, the first term of the bound (6) is unavoidable, and so we can restrict our attention to obtaining good permutation estimates \((\hat{\pi}, \hat{\sigma})\).

Past work [SBW16a, CM16] has typically proceeded from equation (6) by using the inequalities

\[
\frac{1}{n_1 n_2} \|M^*(\pi^*, \sigma^*) - M^*(\hat{\pi}, \hat{\sigma})\|_F^2 \leq \mathcal{R}(M^*, \hat{\pi}), \quad \text{and}
\]

\[
\frac{1}{n_1 n_2} \|M^*(\pi^*, \sigma^*) - M^*(\pi^*, \hat{\sigma})\|_F^2 \leq \mathcal{C}(M^*, \hat{\sigma})
\]

to then reduce the problem to bounding the sum of row-wise and column-wise approximation errors. However, irrespective of how good the permutation estimates \((\hat{\pi}, \hat{\sigma})\) really are, such an analysis approach necessarily produces sub-optimal rates, owing to the lower bounds (5a) and (5b). In particular, given \( N = n^2 \) noisy observations, any such analysis cannot improve upon the Frobenius error rate \( n^{-1/2} \) for \( n \times n \) matrices in the class \( \mathcal{C}_{\text{Perm}}^{r,c} \), or the Frobenius error rate \( n^{-3/4} \) for \( n \times n \) matrices in the class \( \mathcal{C}_{\text{Perm}} \). Consequently, our algorithm for the class \( \mathcal{C}_{\text{Perm}}^{r,c} \) (defined in Section 3.2.2 to follow) exploits a finer analysis technique for the approximation error terms so as to guarantee faster rates.

We now present two permutation estimation procedures that can be plugged into Step 1 of this meta-algorithm.
3.2.1 Matrices with ordered columns

As a stepping stone to our main algorithm, which estimates over the class $C_{\text{Perm}}^r$, we first consider the estimation problem when the permutation along one of the dimensions is known. This corresponds to estimation over the subclass $C_{\text{Perm}}^r$, and following the meta-algorithm above, it suffices to provide a permutation estimate $\hat{\pi}$. The result of this section holds without the assumption $n_1 \geq n_2$.

We need more notation to facilitate the description of the algorithm. We say that $\mathbf{bl} = \{B_k\}_{k=1}^{\lvert \mathbf{bl} \rvert}$ is a partition of $[n_2]$, if $[n_2] = \bigcup_{k=1}^{\lvert \mathbf{bl} \rvert} B_k$ and $B_j \cap B_k = \emptyset$ for $j \neq k$. Moreover, we group the columns of a matrix $Y \in \mathbb{R}^{n_1 \times n_2}$ into $\lvert \mathbf{bl} \rvert$ blocks according to their indices in $\mathbf{bl}$, and refer to $\mathbf{bl}$ as a partition or blocking of the columns of $Y$. In the algorithm, partial row sums of $Y$ are computed on indices contained in each block.

Algorithm 2 (sorting partial sums)

- Step 1: Choose a partition $\mathbf{bl}_{\text{ref}}$ of the set $[n_2]$ consisting of contiguous blocks, such that each block $B$ in $\mathbf{bl}_{\text{ref}}$ has size

$$\frac{1}{2} n_2 \sqrt{\frac{n_1 n_2}{N} \log(n_1 n_2)} \leq |B| \leq n_2 \sqrt{\frac{n_1}{N} \log(n_1 n_2)}.$$

- Step 2: Given the observation matrix $Y$, compute the row sums

$$S(i) = \sum_{j \in [n_2]} Y_{i,j} \quad \text{for each } i \in [n_1],$$

and the partial row sums within each block

$$S_B(i) = \sum_{j \in B} Y_{i,j} \quad \text{for each } i \in [n_1] \text{ and } B \in \mathbf{bl}_{\text{ref}}.$$

Create a directed graph $G$ with vertex set $[n_1]$, where an edge $u \rightarrow v$ is present if either

$$S(v) - S(u) > 16(\zeta + 1) \left( \sqrt{\frac{n_1 n_2}{N} \log(n_1 n_2)} + \frac{n_1 n_2}{N} \log(n_1 n_2) \right), \quad \text{or}$$

$$S_B(v) - S_B(u) > 16(\zeta + 1) \left( \sqrt{\frac{n_1 n_2}{N} \lvert B \rvert \log(n_1 n_2)} + \frac{n_1 n_2}{N} \log(n_1 n_2) \right)$$

for some $B \in \mathbf{bl}_{\text{ref}}$.

- Step 3: Compute a topological sort $\hat{\pi}_{\text{ref}}$ of the graph $G$, and return it as the permutation estimate; if none exists, return $\hat{\pi}_{\text{ref}} = \text{id}$.

Recall that a permutation $\pi$ is called a topological sort of $G$ if $\pi(u) < \pi(v)$ for every directed edge $u \rightarrow v$. The complexity of the algorithm is dominated by Step 2, in which constructing the graph $G$ takes time $O(n_1^2 n_2^{1/2})$, since there are at most $O(n_1^{1/2})$ blocks.

While algorithms in past work [SBW16a, CM16, PMM+17] sort the rows of the matrix according to the full Borda counts $S(i)$ defined in step 2, they are limited by the high standard deviation in these estimates. Our key observation is that when the columns are perfectly ordered, judiciously
chosen partial row sums (which are less noisy than full row sums) also contain information that can help estimate the underlying row permutation. The thresholds on the score differences in step 2 are chosen to be comparable to the standard deviations of the respective estimates, with additional logarithmic factors that allow for high-probability statements via application of Bernstein bounds.

We now characterize the rate achieved by the estimator \( \hat{M}(\tilde{\pi}_{\text{ref}}, \text{id}) \).

**Theorem 3.** For any matrix \( M^* \in \mathbb{C}_{\text{Perm}} \), we have

\[
R(M^*, \tilde{\pi}_{\text{ref}}) \lesssim (\zeta^2 \vee 1) \left( \frac{n_1 \log n_1}{N} \right)^{3/4}
\]

with probability at least \( 1 - 2(n_1 n_2)^{-2} \). Consequently, we also have

\[
F(M^*, \hat{M}(\tilde{\pi}_{\text{ref}}, \text{id})) \lesssim (\zeta^2 \vee 1) \left[ \left( \frac{n_1 \log n_1}{N} \right)^{3/4} + \frac{n_1 \log^2 n_1}{N} \right]
\]

with probability at least \( 1 - 3(n_1 n_2)^{-2} \).

Comparing the bounds (5b) and (7a), we see that the estimator \( \hat{M}(\tilde{\pi}_{\text{ref}}, \text{id}) \), which can be computed in polynomial time, is minimax optimal for the class \( \mathbb{C}_{\text{Perm}} \) (up to logarithmic factors) in the metric \( R(M^*, \tilde{\pi}) \). However, comparing the bounds (4b) and (7b), we see that the estimator falls short of being optimal in the Frobenius error metric. Closing this gap in the Frobenius error uniformly over the class \( \mathbb{C}_{\text{Perm}} \) is an interesting open problem. We now turn to providing estimators for matrices in the class \( \mathbb{C}_{\text{Perm}} \).

### 3.2.2 Two-dimensional sorting for class \( \mathbb{C}_{\text{Perm}} \)

The algorithm in the previous section cannot be immediately extended to the class \( \mathbb{C}_{\text{Perm}} \), since it assumes that the matrix is perfectly sorted along one of the dimensions. However, it suggests a plug-in procedure that can be described informally as follows.

1. Sort the columns of the matrix \( Y \) according to its column sums.
2. Apply Algorithm 2 to the column-sorted matrix to obtain a row permutation estimate.
3. Repeat Steps 1 and 2 with \( Y \) transposed to obtain a column permutation estimate.

Although the columns of \( Y \) are only approximately sorted in the first step, the hope is that the finer row-wise control given by Algorithm 2 is able to improve the row permutation estimate. The actual algorithm, provided below, essentially implements this intuition, but with a careful data-dependent blocking procedure that we describe next. Given a data matrix \( Y \in \mathbb{R}^{n_1 \times n_2} \), the following blocking subroutine returns a column partition \( \mathcal{B}(Y) \).

**Subroutine 1 (blocking)**

- **Step 1:** Compute the column sums \( \{C(j)\}_{j=1}^{n_2} \) of the matrix \( Y \) as

\[
C(j) = \sum_{i=1}^{n_1} Y_{i,j}.
\]

Let \( \tilde{\sigma}_{\text{pre}} \) be a permutation along which the sequence \( \{C(\tilde{\sigma}_{\text{pre}}(j))\}_{j=1}^{n_2} \) is nondecreasing.
• Step 2: Set \( \tau = 16(\zeta + 1)\left(\frac{n_1^2 n_2}{N} \log(n_1 n_2) + \frac{n_2}{\sqrt{N}} \log(n_1 n_2)\right) \) and \( K = \lceil n_2/\tau \rceil \). Partition the columns of \( Y \) into \( K \) blocks by defining

\[
\begin{align*}
bl_1 &= \{ j \in [n_2] : C(j) \in (-\infty, \tau) \}, \\
bl_k &= \{ j \in [n_2] : C(j) \in [(k-1)\tau, k\tau) \} \text{ for } 1 < k < K, \text{ and} \\
bl_K &= \{ j \in [n_2] : C(j) \in [(K-1)\tau, \infty) \}.
\end{align*}
\]

Note that each block is contiguous when the columns are permuted by \( \hat{\sigma}_{\text{pre}} \).

• Step 3 (aggregation): Set \( \beta = n_2^{\frac{1}{2}} \sqrt{n_1 N \log(n_1 n_2)} \). Call a block \( bl_k \) “large” if \( |bl_k| \geq \beta \) and “small” otherwise. Aggregate small blocks in \( bl \) while leaving the large blocks as they are, to obtain the final partition \( BL \).

More precisely, consider the matrix \( Y' = Y(\text{id}, \hat{\sigma}_{\text{pre}}) \) having nondecreasing column sums and contiguous blocks. Call two small blocks “adjacent” if there is no other small block between them. Take unions of adjacent small blocks to ensure that the size of each resulting block is in the range \( \left[ \frac{1}{2} \beta, 2\beta \right] \). If the union of all small blocks is smaller than \( \frac{1}{2} \beta \), aggregate them all.

Return the resulting partition \( BL(Y) = BL \).

Ignoring Step 3 for the moment, we see that the blocking \( bl \) is analogous to the blocking \( bl_{\text{ref}} \) of Algorithm 2, along which partial row-sums may be computed. While the blocking \( bl_{\text{ref}} \) was chosen in a data-independent manner due to the columns being sorted exactly, the blocking \( bl \) is chosen based on approximate estimation of the column permutation. However, some of these \( K \) blocks may be too small, resulting in noisy partial sums; in order to mitigate this issue, Step 3 aggregates small blocks into large enough ones. We are now in a position to describe the two-dimensional sorting algorithm.

Algorithm 3 (two-dimensional sorting)

• Step 0: Split the observations into two independent subsamples of equal size, and form the corresponding matrices \( Y^{(1)} \) and \( Y^{(2)} \) according to equation (3).

• Step 1: Apply Subroutine 1 to the matrix \( Y^{(1)} \) to obtain a partition \( BL = BL(Y^{(1)}) \) of the columns. Let \( K \) be the number of blocks in \( BL \).

• Step 2: Using the second sample \( Y^{(2)} \), compute the row sums

\[
S(i) = \sum_{j \in [n_2]} Y_{i,j}^{(2)} \text{ for each } i \in [n_1],
\]

and the partial row sums within each block

\[
S_{BL_k}(i) = \sum_{j \in BL_k} Y_{i,j}^{(2)} \text{ for each } i \in [n_1], k \in [K].
\]
Create a directed graph $G$ with vertex set $[n_1]$, where an edge $u \to v$ is present if either

$$S(v) - S(u) > 16(\zeta + 1) \left( \frac{n_1 n_2^2}{N} \log(n_1 n_2) + \frac{n_1 n_2}{N} \log(n_1 n_2) \right), \quad \text{or} \quad (8a)$$

$$S_{BL_k}(v) - S_{BL_k}(u) > 16(\zeta + 1) \left( \frac{n_1 n_2^2}{N} |BL_k| \log(n_1 n_2) + \frac{n_1 n_2}{N} \log(n_1 n_2) \right) \quad \text{for some } k \in [K]. \quad (8b)$$

- Step 3: Compute a topological sort $\hat{\pi}_{tds}$ of the graph $G$; if none exists, set $\hat{\pi}_{tds} = \text{id}$.
- Step 4: Repeat Steps 1–3 with $(Y^{(i)} \uparrow)$ replacing $Y^{(i)}$ for $i = 1, 2$, the roles of $n_1$ and $n_2$ switched, and the roles of $\pi$ and $\sigma$ switched, to compute the permutation estimate $\hat{\sigma}_{tds}$.
- Step 5: Return the permutation estimates $(\hat{\pi}_{tds}, \hat{\sigma}_{tds})$.

As before, the construction of the graph $G$ in Step 2 dominates the computational complexity, and takes time $O(n_1^2 n_2/\beta) = O(n_1^2 n_2^{1/2})$. Computing judiciously chosen partial row-sums once again capture a lot more of the signal in the problem, and we have the following guarantee.

**Theorem 4.** For any matrix $M^* \in \mathbb{C}_{\text{Perm}}^{r,c}$, we have

$$\mathcal{R}(M^*, \hat{\pi}_{tds}) \lesssim (\zeta^2 \lor 1) \left( \frac{n_1 \log n_1}{N} \right)^{1/2}, \quad \text{and} \quad (9a)$$

$$\mathcal{F}(M^*, \hat{M}(\hat{\pi}_{tds}, \hat{\sigma}_{tds})) \lesssim (\zeta^2 \lor 1) \left[ \left( \frac{n_1 \log n_1}{N} \right)^{3/4} + \frac{n_1 \log^2 n_1}{N} \right] \quad (9b)$$

with probability at least $1 - 9(n_1 n_2)^{-3}$.

As mentioned before, our estimator is minimax optimal (up to a logarithmic factor) in the row-approximation metric, but this was already achieved by other estimators in the literature [SBW16a, CM16]. The main contribution of our paper is the Frobenius norm guarantee, which breaks a conjectured statistical-computational barrier. In particular, setting $N = n_1 n_2$, we have proved that our efficient estimator enjoys the rate

$$\frac{1}{n_1 n_2} \left\| \hat{M}(\hat{\pi}_{tds}, \hat{\sigma}_{tds}) - M^* \right\|_F^2 = \tilde{O} \left( n_2^{-3/4} \right).$$

It is also worth mentioning that this result extends to estimation of matrices in the class $\mathbb{C}_{\text{SST}}(n)$. In particular, we have $n_1 = n_2 = n$, and either of the two estimates $\hat{\pi}_{tds}$ or $\hat{\sigma}_{tds}$ may be returned as an estimate of the permutation $\pi$. Consequently, the informal theorem stated in the introduction is an immediate corollary of Theorem 4 once these modifications are made to the algorithm.

## 4 Proofs

We now turn to the proofs of our results. Throughout the proofs, we assume without loss of generality that $M^* \in \mathbb{C}_{\text{BISO}}(\text{id}, \text{id}) = \mathbb{C}_{\text{BISO}}$. Because we are interested in rates of estimation up to universal constants, we assume that each independent subsample contains $N' = \text{Poi}(N)$ observations (instead of $\text{Poi}(N)/2$ or $\text{Poi}(N)/4$). We use the shorthand $Y = Y \left( \{(X_\ell, y_\ell)\}_{\ell=1}^{N'} \right)$, throughout.
Some preliminary lemmas

Before turning to proofs of our theorems, we provide three lemmas that underlie many of our arguments. The first lemma can be readily distilled from the proof of Theorem 5 of Shah et al. [SBGW17] with slight modifications. It is worth mentioning that similar lemmas characterizing the estimation error of a bivariate isotonic matrix were also proved by [CGS18, CM16].

**Lemma 1 ([SBGW17]).** Let \( n_1 \geq n_2 \), and let \( M^* \in \mathbb{C}^{r,c}_{\text{Perm}} \). Assume that our observation model takes the form \( Y = M^* + W \), where the noise matrix \( W \) satisfies the properties

(a) the entries \( W_{i,j} \) are independent, centered, \( \frac{c_1}{p_{\text{obs}}} (\zeta \lor 1) \)-sub-Gaussian random variables;

(b) the second moments are bounded as \( \mathbb{E}[|W_{i,j}|^2] \leq \frac{c_2}{p_{\text{obs}}} (\zeta^2 \lor 1) \) for all \( i \in [n_1], j \in [n_2] \).

Then the least squares estimator \( \hat{M}_{\text{LS}}(\mathbb{C}^{r,c}_{\text{Perm}},Y) \) satisfies

\[
\mathbb{P}\left\{ \|\hat{M}_{\text{LS}}(\mathbb{C}^{r,c}_{\text{Perm}},Y) - M^*\|_F^2 \geq \frac{c_3}{p_{\text{obs}}} (\zeta^2 \lor 1)n_1 \log n_1 \right\} \leq (n_1n_2)^{-3}.
\]

Moreover, the same result holds if the class \( \mathbb{C}^{r,c}_{\text{Perm}} \) is replaced by the class \( \mathbb{C}_{\text{BISO}} \).

The proof follows that of Shah et al. [SBGW17, Theorem 5] very closely, and is postponed to Appendix A.1. The next lemma establishes concentration of sums of our observations around their means.

**Lemma 2.** For any nonempty subset \( S \subset [n_1] \times [n_2] \), it holds that

\[
\mathbb{P}\left\{ \left| \sum_{(i,j) \in S} (Y_{i,j} - M^*_{i,j}) \right| \geq 8(\zeta + 1) \left( \sqrt{\frac{|S|n_1n_2}{N}} \log(n_1n_2) + 2 \frac{n_1n_2}{N} \log(n_1n_2) \right) \right\} \leq 2(n_1n_2)^{-4}.
\]

**Proof.** According to definitions (1) and (3), we have

\[
W_{i,j} = Y_{i,j} - M^*_{i,j} = \begin{cases} -M^*_{i,j} & \text{if entry } (i, j) \text{ is not observed,} \\ M^*_{i,j}/p_{\text{obs}} - M^*_{i,j} + \frac{W'}{p_{\text{obs}}} & \text{otherwise,} \end{cases}
\]

where \( W' \) is a \( \zeta \)-sub-Gaussian noise matrix with independent entries. Consequently, we can express the noise on each entry as \( W_{i,j} = Z_{i,j}^{(1)} + Z_{i,j}^{(2)} \) where \( \{ Z_{i,j}^{(1)} \}_{i \in [n_1], j \in [n_2]} \) are independent, zero-mean random variables given by

\[
Z_{i,j}^{(1)} = \begin{cases} M^*_{i,j} (p_{\text{obs}}^{-1} - 1) & \text{with probability } p_{\text{obs}}, \\ -M^*_{i,j} & \text{with probability } 1 - p_{\text{obs}}, \end{cases}
\]

and \( \{ Z_{i,j}^{(2)} \}_{i \in [n_1], j \in [n_2]} \) are independent, zero-mean random variables such that

\[
Z_{i,j}^{(2)} \text{ is } \begin{cases} \zeta \frac{p_{\text{obs}}}{p_{\text{obs}}^2} & \text{with probability } p_{\text{obs}}, \\ 0 & \text{with probability } 1 - p_{\text{obs}}, \end{cases}
\]
We control the two separately. First, we have $|Z_{i,j}^{(1)}| \leq 1/p_{\text{obs}}$ and the variance of each $Z_{i,j}^{(1)}$ is bounded by $(1 - p_{\text{obs}})^2/p_{\text{obs}} + (1 - p_{\text{obs}}) \leq 1/p_{\text{obs}}$. Hence Bernstein’s inequality for bounded noise yields
\[
\Pr \left\{ \left| \sum_{(i,j) \in S} Z_{i,j}^{(1)} \right| \geq t \right\} \leq 2 \exp \left( - \frac{t^2/2}{|S|/p_{\text{obs}} + t/(3p_{\text{obs}})} \right).
\]

Taking $t = 4 \sqrt{|S|/n_1 n_2} \log(n_1 n_2) + 6 n_1 n_2 \log(n_1 n_2)$ and recalling that $p_{\text{obs}} \geq N/2n_1 n_2$, we obtain
\[
\Pr \left\{ \left| \sum_{(i,j) \in S} Z_{i,j}^{(1)} \right| \geq 4 \sqrt{|S|/n_1 n_2} \log(n_1 n_2) + 6 n_1 n_2 \log(n_1 n_2) \right\} \leq (n_1 n_2)^{-4}.
\]

In order to control the deviation of the sum of $Z_{i,j}^{(2)}$, we note that the $q$-th moment of $Z_{i,j}^{(2)}$ is bounded by $N/n_1 n_2 (2/p_{\text{obs}})^q \leq \frac{q^2}{2} 8\zeta n_1 n_2 (4n_1 n_2/N)^q$. Then another version of Bernstein’s inequality [BLM13] yields
\[
\Pr \left\{ \left| \sum_{(i,j) \in S} Z_{i,j}^{(2)} \right| \geq \sqrt{16\zeta^2 |S| n_1 n_2 t} + \frac{4\zeta n_1 n_2 t}{N} \right\} \leq 2 \exp(-t),
\]

and setting $t = 4 \log(n_1 n_2)$ gives
\[
\Pr \left\{ \left| \sum_{(i,j) \in S} Z_{i,j}^{(2)} \right| \geq 8\zeta \sqrt{|S|/n_1 n_2} \log(n_1 n_2) + 16\zeta n_1 n_2 \log(n_1 n_2) \right\} \leq (n_1 n_2)^{-4}.
\]

Combining the above two deviation bounds completes the proof.

The last lemma is a deterministic result.

**Lemma 3.** Let $\{a_i\}_{i=1}^n$ be a nondecreasing sequence of real numbers. If $\pi$ is a permutation in $\mathcal{S}_n$ such that $\pi(i) < \pi(j)$ whenever $a_j - a_i > \tau$ where $\tau > 0$, then $|a_{\pi(i)} - a_i| \leq \tau$ for all $i \in [n]$.

**Proof.** Suppose that $a_j - a_{\pi(j)} > \tau$ for some index $j \in [n]$. Since $\pi$ is a bijection, there must exist an index $i \leq \pi(j)$ such that $\pi(i) > \pi(j)$. However, we then have $a_j - a_i \geq a_j - a_{\pi(j)} > \tau$, which contradicts the assumption. A similar argument shows that $a_{\pi(j)} - a_j > \tau$ also leads to a contradiction. Therefore, we obtain that $|a_{\pi(j)} - a_j| \leq \tau$ for every $j \in [n]$.

With these lemmas in hand, we are now ready to prove our main theorems.

### 4.2 Proof of Theorem 1

We split the proof into two parts by proving the upper and lower bounds separately.
4.2.1 Proof of upper bound

The upper bound follows from Lemma 1 once we check the conditions on the noise for our model. We have seen in the proof of Lemma 2 that the noise on each entry can be written as \( W_{i,j} = Z_{i,j}^{(1)} + Z_{i,j}^{(2)} \). Again, \( Z_{i,j}^{(1)} \) and \( Z_{i,j}^{(2)} \) are \( \frac{c}{p_{\text{obs}}} \)-sub-Gaussian and \( \frac{c_\zeta}{p_{\text{obs}}} \)-sub-Gaussian respectively, and have variances bounded by \( \frac{1}{p_{\text{obs}}} \) and \( \frac{c_\zeta^2}{p_{\text{obs}}} \) respectively. Hence the conditions on \( W \) in Lemma 1 are satisfied. Then we can apply the lemma, recall the relation \( p_{\text{obs}} \geq \frac{N^2 n_1 n_2}{n_1 n_2} \) and normalize the bound by \( \frac{1}{n_1 n_2} \) to complete the proof.

4.2.2 Proof of lower bound

The lower bound follows from an application of Fano’s lemma. The technique is standard, and we briefly review it here. Suppose we wish to estimate a parameter \( \theta \) over an indexed class of distributions \( P = \{ P_\theta \mid \theta \in \Theta \} \) in the square of a (pseudo-)metric \( \rho \). We refer to a subset of parameters \( \{ \theta_1, \theta_2, \ldots, \theta_K \} \) as a local \((\delta, \epsilon)\)-packing set if

\[
\min_{i,j \in [K], i \neq j} \rho(\theta_i, \theta_j) \geq \delta \quad \text{and} \quad \frac{1}{K(K-1)} \sum_{i,j \in [K], i \neq j} D(P_{\theta_i} \| P_{\theta_j}) \leq \epsilon.
\]

Note that this set is a \( \delta \)-packing in the metric \( \rho \) with the average KL-divergence bounded by \( \epsilon \). The following result is a straightforward consequence of Fano’s inequality (see [Tsy09, Theorem 2.5]):

**Lemma 4** (Local packing Fano lower bound). For any \((\delta, \epsilon)\)-packing set of cardinality \( K \), we have

\[
\inf_{\hat{\theta}} \sup_{\theta^* \in \Theta} \mathbb{E} \left[ \rho(\hat{\theta}, \theta^*)^2 \right] \geq \frac{\delta^2}{2} \left( 1 - \frac{\epsilon + \log 2}{\log K} \right).
\]

In addition, the Gilbert-Varshamov bound [Gil52, Var57] guarantees the existence of binary vectors \( \{ v^1, v^2, \ldots, v^K \} \subseteq \{0,1\}^{n_1} \) such that \( K \geq 2^{c_1 n_1} \) and \( \| v^i - v^j \|_2^2 \geq c_2 n_1 \) for each \( i \neq j \),

\[
K \geq 2^{c_1 n_1} \quad \text{and} \quad \| v^i - v^j \|_2^2 \geq c_2 n_1 \quad \text{for each} \quad i \neq j,
\]

for some fixed tuple of constants \( (c_1, c_2) \). We use this guarantee to design a packing of matrices in the class \( \mathbb{C}_{\text{Perm}}^r \). For each \( i \in [K] \), fix some \( \delta \in [0, 1/4] \) to be precisely set later, and define the matrix \( M^i \) having identical columns, with entries given by

\[
M^i_{j,k} = \begin{cases} 
1/2, & \text{if } v^i_j = 0 \\
1/2 + \delta, & \text{otherwise}. 
\end{cases}
\]

Clearly, each of these matrices \( \{ M^i \}_{i=1}^K \) is a member of the class \( \mathbb{C}_{\text{Perm}}^r \), and each distinct pair of matrices \( (M^i, M^j) \) satisfies the inequality \( \| M^i - M^j \|_F^2 \geq c_2 n_1 n_2 \delta^2 \).

Let \( P_M \) denote the probability distribution of the observations in the model (1) with underlying matrix \( M \in \mathbb{C}_{\text{Perm}}^r \). Our observations are independent across entries of the matrix, and so the KL divergence tensorizes to yield

\[
D(P_{M^i} \| P_{M^j}) = \sum_{k \in [n_1]} D(P_{M^i_{k,\ell}} \| P_{M^j_{k,\ell}}). 
\]
Let us now examine one term of this sum. Note that we observe \( \kappa_{k,\ell} \sim \text{Poi}\left(\frac{N}{n_1 n_2}\right) \) samples of each entry \((k,\ell)\).

Under the Bernoulli observation model (2b), conditioned on the event \( \kappa_{k,\ell} = \kappa \), we have the distributions
\[
P_{M_{k,\ell}^i} = \text{Bin}(\kappa, M_{k,\ell}^i), \quad \text{and} \quad P_{M_{k,\ell}^j} = \text{Bin}(\kappa, M_{k,\ell}^j).
\]

Consequently, the KL divergence conditioned on \( \kappa_{k,\ell} = \kappa \) is given by
\[
D(P_{M_{k,\ell}^i} \parallel P_{M_{k,\ell}^j}) = \kappa D(M_{k,\ell}^i \parallel M_{k,\ell}^j),
\]
where we have used \( D(p \parallel q) = p \log(p) + (1-p) \log(1-q) \) to denote the KL divergence between the Bernoulli random variables \( \text{Ber}(p) \) and \( \text{Ber}(q) \).

Note that for \( p, q \in [1/2, 3/4] \), we have
\[
D(p \parallel q) = p \log\left(\frac{p}{q}\right) + (1-p) \log\left(\frac{1-p}{1-q}\right) \leq (i) \left( \frac{p-q}{q} \right)^2 = p \log\left(\frac{p}{q}\right) + (1-p) \left( \frac{1-p}{1-q} \right) = \frac{16}{3} (p-q)^2.
\]

Taking the expectation over \( \kappa \), we have
\[
D(P_{M_{k,\ell}^i} \parallel P_{M_{k,\ell}^j}) \leq \frac{16}{3} N \frac{n_1 n_2}{N} \left( M_{k,\ell}^i - M_{k,\ell}^j \right)^2 \leq \frac{16}{3} N \frac{n_1 n_2}{n_1 n_2} \delta^2,
\]
Summing over \( k \in [n_1], \ell \in [n_2] \) yields \( D(P_{M^i} \parallel P_{M^j}) \leq \frac{16}{3} N \delta^2 \).

Under the standard Gaussian observation model (2a), a similar argument yields the bound \( D(P_{M^i} \parallel P_{M^j}) \leq \frac{1}{2} N \delta^2 \), since we have \( D(\mathcal{N}(p,1) \parallel \mathcal{N}(q,1)) = (p-q)^2/2 \).

Substituting into Fano’s inequality (10), we have
\[
\inf_{\hat{M}} \sup_{M^* \in \mathcal{C}_{\text{perm}}} \mathbb{E}\left[ \| \hat{M} - M^* \|^2_F \right] \geq \frac{c_2 n_1 n_2 \delta^2}{2} \left( 1 - \frac{16}{3} N \delta^2 + \log 2 \right).
\]

Finally, choosing \( \delta^2 = c_3 n_1 \frac{N}{N} \) and normalizing by \( n_1 n_2 \) yields the claim.

### 4.3 Proof of Theorem 2

We prove the theorem for the observation model with standard Gaussian noise. The proof for Bernoulli observations is analogous, but we defer it to Appendix A.2. (It proceeds in much the same fashion, but requires a rather technical lemma.)

At a high level, our strategy is to reduce the problem to that of hypothesis testing over particular instances of convex cones; we note that cone testing problems have been extensively studied in past...
work (see the paper [WWG17] and references therein). The reduction takes the following form. Consider the subclasses of $C_{\text{Perm}}^r$ and $C_{\text{Perm}}^{r,c}$ in which the first $n_1 - 2$ rows are identically zero. In this special case, obtaining a good estimate of the row permutation $\hat{\pi}$ in the metric $\mathcal{R}(M^*, \hat{\pi})$ corresponds to correctly ordering the last two rows of the matrix. This ordering problem can be reduced to a compound hypothesis testing problem.

Before diving into the details, let us introduce some useful notation. Recall that in our model, the number of observations at each entry has distribution $\text{Poi}(\lambda)$ where we set $\lambda := \frac{N}{n_1 n_2}$. For a vector $x \in [0, 1]^{n_2}$, suppose that at each entry $i$, we observe $\kappa_i \sim \text{Poi}(\lambda)$ noisy samples of $x_i$. Under the Gaussian observation model, the distribution $\mathbb{G}(x)$ of the observations, when conditioned on a particular realization of $\kappa = (\kappa_1, \ldots, \kappa_{n_2})$, takes the form

$$\mathbb{G}(x; \kappa) := \bigotimes_{i=1}^{n_2} \bigotimes_{\ell=1}^{\kappa_i} \mathcal{N}(x_i, 1),$$

(13)

where $\mathcal{N}(c, 1)$ denotes the standard Gaussian distribution with mean $c$.

We now set up a precise testing problem. For a set $S \subseteq [0, 1]^{n_2} \times [0, 1]^{n_2}$ of pairs of vectors $(u, v)$ and a radius of testing $r$ to be chosen, let $\mathbb{M}_S(r)$ denote a mixture distribution supported on vector pairs $(u, v) \in S$ that obey the conditions

$$u \preceq v, \quad \text{and} \quad \|u - v\|_2 \geq r.$$

(14a)

(14b)

Consider the testing problem based on the pair of compound hypotheses

- $H_0$: $y_1 \sim \mathbb{G}(u)$ and $y_2 \sim \mathbb{G}(v)$ where $(u, v) \sim \mathbb{M}_S(r)$, and
- $H_1$: $y_1 \sim \mathbb{G}(v)$ and $y_2 \sim \mathbb{G}(u)$ where $(u, v) \sim \mathbb{M}_S(r)$.

Denote by $\mathbb{P}_0$ and $\mathbb{P}_1$ the distributions of the pair $(y_1, y_2)$ under the hypotheses $H_0$ and $H_1$, respectively. We will construct a mixture distribution $\mathbb{M}_S(r)$ such that

$$\text{TV}(\mathbb{P}_0, \mathbb{P}_1) \leq \frac{1}{2}.$$  

(14c)

The following lemma shows that in order to establish a lower bound in the $\mathcal{R}$ metric, it is sufficient to lower bound the minimax testing radius of the hypothesis testing problem above for different choices of the set $S$.

**Lemma 5.** (a) Let $S$ be the set of pairs of vectors $(u, v)$ in $[0, 1]^{n_2}$ such that there is a common permutation $\pi$ for which $\{u_{\pi(i)}\}_{i=1}^{n_2}$ and $\{v_{\pi(i)}\}_{i=1}^{n_2}$ are both nondecreasing. Suppose that there exists a mixture $\mathbb{M}_S(r)$ and associated observation distributions $\mathbb{P}_0$ and $\mathbb{P}_1$ that obey the conditions (14a), (14b), and (14c). Then we have

$$\inf_{\hat{\pi}} \sup_{M^* \in C_{\text{Perm}}^{r,c}} \mathbb{E}[\mathcal{R}(M^*, \hat{\pi})] \geq \frac{r^2}{4n_2}.$$  

(b) Suppose that the conditions above hold with $S$ being the set of pairs of nondecreasing vectors in $[0, 1]^{n_2}$. Then we have

$$\inf_{\hat{\pi}} \sup_{M^* \in C_{\text{Perm}}} \mathbb{E}[\mathcal{R}(M^*, \hat{\pi})] \geq \frac{r^2}{4n_2}.$$  


Thus, in order to prove the theorem, it suffices to construct a mixture distribution $M_S$ with the required properties. The following technical lemma provides bounds on the KL divergence between mixture distributions that fit our needs.

**Lemma 6.** Let $\{\theta^0, \theta^1, \ldots, \theta^s\}$ denote $s + 1$ vectors in $[1/4, 3/4]^d$. Choose indices $\alpha$ and $\beta$ uniformly at random from the sets $[s]$ and $\{0, 1, \ldots, s - 1\}$ respectively. If $P = \mathbb{G}(\theta^\alpha)$ and $Q = \mathbb{G}(\theta^\beta)$, then

$$\text{KL}(P, Q) \leq s^{-1} \left( 2.1 + \lambda \|\theta^\alpha - \theta^{s-1}\|_2^2 + \sqrt{\lambda} \|\theta^s - \theta^{s-1}\|_2 \right).$$

Delaying the proofs of Lemmas 5 and 6 to later, we are now ready to show the existence of mixtures satisfying the conditions (14a), (14b), and (14c) in order to prove each part of our theorem.

### 4.3.1 Proof of part (a)

Define the triplet of scalars

$$s = 6(n_2 \lambda)^{1/2}, \quad t = \lambda^{-1}, \quad \text{and} \quad r = n_2/(st) = \frac{1}{6}(n_2 \lambda)^{1/2} = s/36.$$

We assume that these quantities are integers so as to ease the notation. En route to defining the mixture distribution of interest, we first define random vectors $\mu$ and $\nu$ in $\mathbb{R}^s$ by the assignment

$$\mu_J = 1\{j > J\} \quad \text{and} \quad \nu_J = 1\{j \geq J\},$$

where $J$ is a uniform random index in $[s]$. Note that $\mu$ or $\nu$ is simply a step function on $s$ coordinates, where the location of the jump is random.

Next, we define random vectors $u$ and $v$ on $\mathbb{R}^st$ by “fattening” $\mu$ and $\nu$ respectively. To be more precise, for each $i \in [st]$ define

$$u_i = \frac{1}{2} \left( \mu_{[i/t]} + 1/2 \right) \quad \text{and} \quad v_i = \frac{1}{2} \left( \nu_{[i/t]} + 1/2 \right).$$

In other words, we lift $\mu$ and $\nu$ to $st$ dimensions by replacing each 0 by $t$ copies of 1/4, and each 1 by $t$ copies of 3/4.

Now define random vectors $u$ and $v$ in $\mathbb{R}^{n_2}$ as follows. First we split the $n_2$ coordinates into $r$ consecutive blocks, each of size $st$. Then we set each block of $u$ independently to be equal to $u$ in distribution, and each block of $v$ independently to be equal to $v$ in distribution. Denote by $M_S(r)$ the distribution of the random pair $(u, v)$ constructed above.

Note that there exists a common permutation which reorders the entries of $u$ and $v$ to be nondecreasing: we order the entries equal to 1/4 to be the first, then the entries where “jumps” occur (so that $u$ takes value 1/4 and $v$ takes value 3/4 on those entries), and finally the entries equal to 3/4. Therefore, the assumption in Lemma 5 is satisfied. We claim that the mixture $M_S(r)$ and the pair of distributions $(P_0, P_1)$ satisfy conditions (14a), (14b), and (14c).

Condition (14a) holds by definition. Condition (14b) holds with $r^2 \gtrsim (n_2/\lambda)^{1/2}$, since the vectors $u$ and $v$ differ by 1/2 on $t = \lambda^{-1}$ entries of each block, and there are $r \gtrsim (n_2 \lambda)^{1/2}$ blocks in total. Hence the rate given by Lemma 5 is $r^2/n_2 \gtrsim (\lambda n_2)^{-1/2} = (n_1/N)^{1/2}$, as desired.
Verifying condition (14c). In order to bound the total variation distance between $P_0$ and $P_1$, we need some more notation. Let $M_1$ and $M_2$ denote the (marginal) distributions of $u$ and $v$ respectively when $(u, v) \sim M_S(r)$. Let $\tilde{P}_0$ and $\tilde{P}_1$ denote the distributions $G(u)$ and $G(v)$ respectively, when $u \sim M_1$ and $v \sim M_2$. Let $P_\ast$ be the distribution of $(y_1^*, y_2^*)$ where $y_1^*, y_2^* \sim G(v)$ independently for $v \sim M_2$. Note that

$$TV(P_0, P_\ast) = TV(\tilde{P}_0, \tilde{P}_1)$$

and

$$TV(P_\ast, P_1) = TV(\tilde{P}_1, \tilde{P}_0)$$

since $P_\ast$ coincides with $P_0$ (or $P_1$) on one component. Then we obtain from the triangle inequality and symmetry of total variation that

$$TV(P_0, P_1) \leq TV(P_0, P_\ast) + TV(P_\ast, P_1) = 2TV(\tilde{P}_0, \tilde{P}_1).$$

Pinsker’s inequality implies that $TV(\tilde{P}_0, \tilde{P}_1) \leq \sqrt{KL(\tilde{P}_0, \tilde{P}_1)/2}$, so it suffices to prove the bound

$$KL(\tilde{P}_0, \tilde{P}_1) \leq 1/8.$$

Note that by the construction of the random vectors $u$ and $v$, sub-vectors of length $st$ are in fact independent across the $r$ blocks. Therefore, we can write the distribution $\tilde{P}_0$ as a product $\tilde{P}_0 = \otimes_{k=1}^r \tilde{P}_0^{(k)}$, where $\tilde{P}_0^{(k)}$ is the distribution of the random sub-vector on the $k$-th block. The analogous statement also holds for $\tilde{P}_1$. It thus suffices to prove the upper bound

$$KL(\tilde{P}_0^{(k)}, \tilde{P}_1^{(k)}) \leq 1/(8r)$$

by virtue of tensorization of the KL divergence.

Let $\theta^j \in \{1/4, 3/4\}^{st}$ denote the vector having $s$ blocks, each of size $t$, with all entries in the first $j$ blocks equal to $1/4$ and all entries in the last $s - j$ blocks equal to $3/4$. Note that $u$ and $v$ are drawn uniformly from the sets $\{\theta^1, \theta^2, \ldots, \theta^s\}$ and $\{\theta^0, \theta^1, \ldots, \theta^{s-1}\}$, respectively. Hence we have $\|\theta^s - \theta^{s-1}\|_2^2 = 1/(4\lambda)$. Applying Lemma 6 yields

$$KL(\tilde{P}_0^{(k)}, \tilde{P}_1^{(k)}) \leq s^{-1} (2.1 + 1/4 + 1/2) \leq 1/(8r),$$

which completes the proof.

4.3.2 Proof of part (b)

The structure of the proof resembles that of part (a), so we focus on the differences. Let

$$s = 6(n_2\lambda)^{1/4}, \quad t = \left(\frac{n_2}{\lambda}\right)^{1/2}, \quad \text{and} \quad r = n/(st) = \frac{1}{6} (n_2\lambda)^{1/4} = s/36.$$

Proceeding as before, we build up to the constructions required to prove the theorem: we first define random vectors $\mu$ and $\nu$ in $\mathbb{R}^s$ by the assignment

$$\mu_j = 1\{j > J\} \quad \text{and} \quad \nu_j = 1\{j \geq J\},$$

where $J$ is a uniform random index in $[s]$. 


Next, we define random vectors \( u \) and \( v \) on \( \mathbb{R}^{st} \) by setting
\[
  u_i = \mu_{[i/t]} \cdot s^{-1} \quad \text{and} \quad v_i = \nu_{[i/t]} \cdot s^{-1}
\]
for each \( i \in [st] \). In contrast to before, we do not shift the vectors, and now scale them by \( s^{-1} \).

Finally, we define random vectors \( u \) and \( v \) in \( \mathbb{R}^{n_2} \) as follows. Split the \( n_2 \) coordinates into \( r \) consecutive blocks, each of size \( st \). Let \( b \) denote the all-ones vector in \( \mathbb{R}^{st} \). For each even index \( k \in [r] \), define the sub-vectors of both \( u \) and \( v \) on the \( k \)-th block to be \( b \cdot (\frac{1}{4} + k s^{-1}) \). On the other hand, for each odd index \( k \) in \([r]\), define the sub-vectors of \( u \) and \( v \) on the \( k \)-th block respectively to be
\[
b \cdot \left( \frac{1}{4} + k s^{-1} \right) + u^{(k)} \quad \text{and} \quad b \cdot \left( \frac{1}{4} + k s^{-1} \right) + v^{(k)},
\]
where each \((u^{(k)}, v^{(k)})\) is an independent copy of the pair of random vectors \((u, v)\). Note that by construction, \( u \) and \( v \) are both nondecreasing vectors in \([1/4, 3/4]^{n_2}\), and we have that \( u \preceq v \).

Denote by \( M_S(r) \) the distribution of the random pair \((u, v)\) constructed above. It suffices to prove that the mixture \( M_S(r) \) and the pair of distributions \((P_0, P_1)\) satisfy conditions (14a), (14b) and (14c).

Condition (14a) holds by definition. Condition (14b) holds with \( r^2 \gtrsim \frac{n_2}{3\eta^2} \), since the vectors \( u \) and \( v \) differ in the squared \( \ell_2 \) norm by \( s^{-2} t \gtrsim \lambda^{-1} \) on each block, and there are \( r \gtrsim (n_2 \lambda)^{1/4} \) blocks in total. The rate given by Lemma 5 is therefore \( \mathcal{R} \gtrsim (\lambda n_2)^{-3/4} = (\frac{n_1}{N})^{3/4} \).

In order to verify condition (14c), note that by the same argument as in part (a), it suffices to prove the bound on the KL divergence
\[
\text{KL}(\widehat{P}_0, \widehat{P}_1) \leq 1/(8r).
\]
Analogously, for \( j \in \{0\} \cup [n] \), we define \( \theta^j \in \{1/4, 3/4\}^{s t} \) to be the vector having \( s \) blocks, each of size \( t \), with all entries in the first \( j \) blocks equal to \( \frac{1}{4} + ks^{-1} \) and all entries in the last \( s - j \) blocks equal to \( \frac{1}{4} + ks^{-1} + s^{-1} \). Then sub-vectors of \( u \) and \( v \) on the \( k \)-block of size \( st \) are distributed uniformly in the sets \( \{\theta^1, \theta^2, \ldots, \theta^{s}\} \) and \( \{\theta^0, \theta^1, \ldots, \theta^{s-1}\} \) respectively. Moreover, we have \( \|\theta^s - \theta^{s-1}\|_2^2 = s^{-2} t = \frac{1}{36x} \). Applying Lemma 6 again yields the above bound.

### 4.3.3 Proof of Lemma 5

We prove part (a) of the lemma; part (b) follows analogously. It suffices to prove lower bounds over procedures provided with the additional information that \( \pi^*(i) = i \) for all \( i \in [n_1 - 2] \). Consider the class of matrices with the first \( n_1 - 2 \) rows set identically to zero. Now choose a vector pair \((u, v) \sim M_S(r)\), and set
\[
\begin{bmatrix}
  M_{n_1-1}^* \\
  M_{n_1}^*
\end{bmatrix}
\]
with prob. \( 1/2 \)

\[
\begin{bmatrix}
  v \\
  u
\end{bmatrix}
\]
with prob. \( 1/2 \).

The minimax rate is clearly lower bounded by the Bayes risk over this random choice of \( M^* \).
Our observations are now given by a noisy version of $M^*$, where each entry of $M^*$ is observed independently a Pois($\frac{N}{m_1 n_2}$) number of times. Let the last two rows of our observations be given by $y_1$ and $y_2$, respectively. Then distinguishing the two cases in equation (16) is equivalent to testing between the hypotheses $H_0$ and $H_1$ defined above. For any pair of permutation estimates $(\hat{\pi}, \hat{\sigma})$, we may thus write

$$
\mathbb{E} \mathcal{R}(M^*, \hat{\pi}) \geq \frac{1}{n_2} \mathbb{E} \left[ \| [M^*(\pi^*, \sigma^*)]_{n_1} - [M^*(\hat{\pi}, \hat{\sigma})]_{n_1} \|^2 \right]
$$

\begin{align*}
&\geq \frac{1}{n_2} \left( \min_{(u,v) \sim \mathcal{M}} \| u - v \|^2_2 \right) \cdot \Pr \{ [\hat{\pi}(n_1 - 1), \hat{\pi}(n_1)] \neq [\pi^*(n_1 - 1), \pi^*(n_1)] \} \\
&\geq \frac{1}{n_2} r^2 \cdot \left( \frac{1}{2} \Pr \{ \phi = 1 \mid H_0 \} + \frac{1}{2} \Pr \{ \phi = 0 \mid H_1 \} \right),
\end{align*}

where in step (i), we have used the fact that for any instance $(u, v)$ from the mixture $\mathbb{M}_S(r)$, it holds that $\min \{ \| u \|^2_2, \| v \|^2_2 \} \geq \| u - v \|^2_2$, so that the returned permutation is always worse off if it swaps the last two rows with any of the rows of $M^*$ that are identically zero. In step (ii), the test $\phi$ distinguishing the hypotheses $H_0$ and $H_1$ is defined by

$$
\phi = \begin{cases} 
0 & \text{if } [\hat{\pi}(n_1 - 1), \hat{\pi}(n_1)] = [n_1 - 1, n_1] \\
1 & \text{if } [\hat{\pi}(n_1 - 1), \hat{\pi}(n_1)] = [n_1, n_1 - 1].
\end{cases}
$$

The probability of error of the test is lower bounded by $\frac{1}{2} (1 - \text{TV}(\mathbb{P}_0, \mathbb{P}_1)) \geq \frac{1}{4}$ by assumption, so this completes the proof.

### 4.3.4 Proof of Lemma 6

Recall that $\kappa = (\kappa_1, \ldots, \kappa_d)$ has i.i.d. entries with distribution Pois($\lambda$). We condition on each instance of $\kappa$ and write the KL divergence as

$$
\text{KL}(\mathbb{P}, \mathbb{Q}) = \mathbb{E}_\mathbb{P} \left[ \log \frac{d\mathbb{P}}{d\mathbb{Q}} \right] = \mathbb{E}_\kappa \mathbb{E}_\mathbb{P} \left[ \log \frac{d\mathbb{P}}{d\mathbb{Q}} \mid \kappa \right].
$$

(17)

For each $j \in \{0\} \cup [s]$, let $p^j$ be the density of $\mathcal{G}(\theta^j; \kappa)$. In other words, if $\mu^j$ denotes the vector in $\mathbb{R}^{\|\kappa\|_1}$ that is the Cartesian product of $\kappa_i$ copies of $\theta^j_i$ over all $i \in [d]$, then $p^j$ is the density of $\mathcal{N}(\mu^j, I_{\|\kappa\|_1})$.

Then we have

\begin{align*}
\mathbb{E}_\kappa \left[ \log \frac{d\mathbb{P}}{d\mathbb{Q}} \mid \kappa \right] &= \int \sum_{j=1}^s s^{-1} p^j \log \frac{\sum_{i=1}^s p^j_{i}}{\sum_{i=0}^{s-1} p^j} \\
&\leq s^{-1} \int \sum_{j=0}^s p^j \log \frac{\sum_{i=0}^s p^{j}_{i}}{\sum_{i=0}^{s-1} p^j} \\
&\leq s^{-1} \int \sum_{j=0}^{s-1} p^j \log \left( 1 + \frac{p^s}{\sum_{i=0}^{s-1} p^j} \right) + s^{-1} \int p^s \log \left( 1 + \frac{p^s}{p^{s-1}} \right) \\
&\leq s^{-1} \int p^s + s^{-1} \int p^s \log \left( 1 + \frac{p^s}{p^{s-1}} \right),
\end{align*}

(18)
where step (i) follows from the inequality \( \log(1 + x) \leq x \). The first term in (18) is simply equal to \( s^{-1} \). To analyze the second term in (18), note that

\[
\int p_s \log \left( 1 + \frac{p_s}{p_s^{s-1}} \right) \leq \int_{p_s^{s-1} \leq 2} p_s \log 3 + 2 \int_{p_s^{s-1} > 2} p_s \log \frac{p_s}{p_s^{s-1}} \leq 1.1 + 2 \int_{p_s^{s-1} > 2} p_s \log \frac{p_s}{p_s^{s-1}}. \tag{19}
\]

By the definition of \( p^j \) and \( \mu^j \), we have

\[
\frac{p^s}{p_s^{s-1}}(y) = \exp \left( -\frac{1}{2} \| y - \mu^s \|^2 + \frac{1}{2} \| y - \mu^{s-1} \|^2 \right) = \exp \left( \frac{1}{2} \| \mu^s - \mu^{s-1} \|^2 + \langle y - \mu^s, \mu^s - \mu^{s-1} \rangle \right). \tag{20}
\]

Define the event

\[
\mathcal{E} := \left\{ \left\langle y - \mu^s, \frac{\mu^s - \mu^{s-1}}{\| \mu^s - \mu^{s-1} \|_2} \right\rangle > \frac{\log 2}{\| \mu^s - \mu^{s-1} \|^2} - \frac{1}{2} \| \mu^s - \mu^{s-1} \|^2 \right\} \overset{(i)}{=} \left\{ \frac{p^s}{p_s^{s-1}}(y) > 2 \right\},
\]

where step (i) holds as a result of equation (20). Therefore, it holds that

\[
\int_{\mathcal{E}} p^s \log \frac{p^s}{p_s^{s-1}} = \int_{\mathcal{E}} p^s(y) \left( \frac{1}{2} \| \mu^s - \mu^{s-1} \|^2 + \langle y - \mu^s, \mu^s - \mu^{s-1} \rangle \right) \leq \frac{1}{2} \| \mu^s - \mu^{s-1} \|^2 + \left( \int_{\mathcal{E}} p^s(y)(y - \mu^s), \mu^s - \mu^{s-1} \right). \tag{21}
\]

Applying Lemma 11 (see the appendix) yields

\[
\int_{\mathcal{E}} p^s(y)(y - \mu^s) = \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{1}{2} \left( \frac{\log 2}{\| \mu^s - \mu^{s-1} \|^2} - \frac{1}{2} \| \mu^s - \mu^{s-1} \|^2 \right)^2 \right) \frac{\| \mu^s - \mu^{s-1} \|^2}{\| \mu^s - \mu^{s-1} \|^2}.
\]

In conjunction with equation (21) and the bound \( e^{-a} \leq 1 \) for \( a \geq 0 \), this yields

\[
\int_{\mathcal{E}} p^s \log \frac{p^s}{p_s^{s-1}} \leq \frac{1}{2} \| \mu^s - \mu^{s-1} \|^2 + \frac{1}{\sqrt{2\pi}} \| \mu^s - \mu^{s-1} \|^2. \tag{22}
\]

Now using the bounds (22), (19), and (18) together yields

\[
\mathbb{E}_\sigma \left[ \log \frac{d\mathbb{P}}{d\mathbb{Q}} \middle| \kappa \right] \leq s^{-1} \left( 2.1 + \| \mu^s - \mu^{s-1} \|^2 + \| \mu^s - \mu^{s-1} \|_2 \right). \tag{23}
\]

Finally, by the definition of \( \mu^j \) we have

\[
\mathbb{E}_\kappa \left[ \| \mu^s - \mu^{s-1} \|^2 \right] = \mathbb{E}_\kappa \left[ \sum_{i=1}^d \kappa_i (\theta_i^s - \theta_i^{s-1})^2 \right] = \lambda \| \theta^s - \theta^{s-1} \|^2,
\]

and substituting into equation (17), we have

\[
\text{KL}(\mathbb{P}, \mathbb{Q}) \leq s^{-1} \left( 2.1 + \mathbb{E}_\kappa \left[ \| \mu^s - \mu^{s-1} \|^2 \right] + \sqrt{\mathbb{E}_\kappa \left[ \| \mu^s - \mu^{s-1} \|^2 \right]} \right) = s^{-1} \left( 2.1 + \lambda \| \theta^s - \theta^{s-1} \|^2 + \sqrt{\lambda} \| \theta^s - \theta^{s-1} \|_2 \right).
\]
4.4 Proof of Proposition 1

Recall the definition of $\hat{M}(\hat{\pi}, \hat{\sigma})$ in the meta-algorithm, and additionally, define the projection of any matrix $M \in \mathbb{R}^{n_1 \times n_2}$, as

$$\mathcal{P}_{\pi, \sigma}(M) = \arg \min_{M \in \mathbb{C}_{\text{BISO}}(\pi, \sigma)} \|M - \hat{M}\|_F.$$ 

and letting $W = Y^{(2)} - M^*$, we have

$$\|\hat{M}(\hat{\pi}, \hat{\sigma}) - M^*\|_F^2 \leq 2\|\mathcal{P}_{\pi, \sigma}(M^* + W) - \mathcal{P}_{\pi, \sigma}(M^*(\hat{\pi}, \hat{\sigma}) + W)\|_F^2 + 2\|\mathcal{P}_{\pi, \sigma}(M^*(\hat{\pi}, \hat{\sigma}) + W) - M^*\|_F^2.$$ 

where step (ii) follows from the non-expansiveness of a projection onto a convex set, and steps (i) and (iii) from the triangle inequality.

The first term in (24) is the estimation error of a bivariate isotonic matrix with known permutations. Since the sample used to obtain $(\hat{\pi}, \hat{\sigma})$ is independent from the sample used in the projection step, it is equivalent to control the error $\|\mathcal{P}_{\text{id}}(M^* + W) - M^*\|_F^2$. As before, the noise matrix $W$ satisfies the conditions of Lemma 1. Therefore, applying Lemma 1 in the case $M^* \in \mathbb{C}_{\text{BISO}}$ with $p_{\text{obs}} \geq \frac{N}{mn_1n_2}$ yields the desired bound of order $(\zeta^2 \vee 1)\frac{n_1 \log^2 n_1}{N}$.

The approximation error can be split into two components: one along the rows of the matrix, and the other along the columns. More explicitly, we have

$$\|M^* - M^*(\hat{\pi}, \hat{\sigma})\|_F^2 \leq 2\|M^* - M^*(\hat{\pi}, \text{id})\|_F^2 + 2\|M^*(\hat{\pi}, \text{id}) - M^*(\hat{\pi}, \hat{\sigma})\|_F^2$$

$$= 2\|M^* - M^*(\hat{\pi}, \text{id})\|_F^2 + 2\|M^* - M^*(\text{id}, \hat{\sigma})\|_F^2.$$ 

The proof readily extends to the general case by precomposing $\hat{\pi}$ and $\hat{\sigma}$ with $\pi^{-1}$ and $\sigma^{-1}$ respectively. This completes the proof of the proposition.

4.5 Proof of Theorem 3

In order to ease the notation, we adopt the shorthand

$$\eta := 16(\zeta + 1)\left(\sqrt{\frac{n_1n_2}{N} \log(n_1n_2)} + 2\frac{n_1n_2}{N} \log(n_1n_2)\right),$$

and for each block $B \in \mathcal{B}_{\text{ref}}$ in Algorithm 2, we use the shorthand

$$\eta_B := 16(\zeta + 1)\left(\sqrt{\frac{|B|n_1n_2}{N} \log(n_1n_2)} + 2\frac{n_1n_2}{N} \log(n_1n_2)\right)$$

throughout the proof. Applying Lemma 2 with $S = \{i\} \times [n_2]$ and then with $S = \{i\} \times B$ for each $i \in [n_1]$ and $B \in \mathcal{B}_{\text{ref}}$, we obtain that

$$\Pr\left\{\left|\sum_{\ell \in [n_2]} M^*_{i, \ell} - \frac{\eta}{2}\right| \geq \frac{\eta}{2}\right\} \leq 2(n_1n_2)^{-4},$$

(25a)
and that
\[
\Pr \left\{ \left| S_B(i) - \sum_{\ell \in B} M_{i,\ell}^* \right| \geq \frac{\eta_B}{2} \right\} \leq 2(n_1n_2)^{-4}. \tag{25b}
\]

A union bound over all rows and blocks yields that \( \Pr \{ \mathcal{E} \} \geq 1 - 2(n_1n_2)^{-3} \), where we define the event
\[
\mathcal{E} := \left\{ \left| S(i) - \sum_{\ell \in [n_2]} M_{i,\ell}^* \right| \leq \frac{\eta}{2} \text{ and } \left| S_B(i) - \sum_{\ell \in B} M_{i,\ell}^* \right| \leq \frac{\eta_B}{2} \text{ for all } i \in [n_1], B \in \text{bl}_{\text{ref}} \right\}.
\]

We now condition on event \( \mathcal{E} \). Applying the triangle inequality yields that if \( S(v) - S(u) > \eta \) or \( S_B(v) - S_B(u) > \eta_B \), then we have
\[
\sum_{\ell \in [n_2]} M_{v,\ell}^* - \sum_{\ell \in [n_2]} M_{u,\ell}^* > 0 \quad \text{or} \quad \sum_{\ell \in B} M_{v,\ell}^* - \sum_{\ell \in B} M_{u,\ell}^* > 0.
\]

It follows that \( u < v \) since \( M^* \) has nondecreasing columns. Thus, by the choice of thresholds \( \eta \) and \( \eta_B \) in the algorithm, we have guaranteed that every edge \( u \to v \) in the graph \( G \) is consistent with the underlying permutation \( \pi \), so a topological sort exists on event \( \mathcal{E} \).

Conversely, if we have
\[
\sum_{\ell \in [n_2]} M_{v,\ell}^* - \sum_{\ell \in [n_2]} M_{u,\ell}^* > 2\eta \quad \text{or} \quad \sum_{\ell \in B} M_{v,\ell}^* - \sum_{\ell \in B} M_{u,\ell}^* > 2\eta_B,
\]
then the triangle inequality implies that
\[
S(v) - S(u) > \eta \quad \text{or} \quad S_B(v) - S_B(u) > \eta_B.
\]

Hence the edge \( u \to v \) is present in the graph \( G \), so the topological sort \( \tilde{\pi}_{\text{ref}}(u) \) satisfies the relation \( \tilde{\pi}_{\text{ref}}(u) < \tilde{\pi}_{\text{ref}}(v) \). Claim that this allows us to obtain the following bounds on event \( \mathcal{E} \):
\[
\left| \sum_{j \in [n_2]} (M_{\tilde{\pi}_{\text{ref}}(i),j}^* - M_{i,j}^*) \right| \leq 96(\zeta + 1) \sqrt{\frac{n_1n_2}{N} \log(n_1n_2)} \quad \text{for all } i \in [n_1], \quad \text{and} \tag{26a}
\]
\[
\left| \sum_{j \in B} (M_{\tilde{\pi}_{\text{ref}}(i),j}^* - M_{i,j}^*) \right| \leq 96(\zeta + 1) \sqrt{\frac{n_1n_2}{N} |B| \log(n_1n_2)} \quad \text{for all } i \in [n_1], B \in \text{bl}_{\text{ref}}. \tag{26b}
\]

We now prove inequality (26b). The proof of inequality (26a) follows in the same fashion. We split the proof into two cases.

**Case 1.** First, suppose that \( |B| \geq \frac{n_1n_2}{N} \log(n_1n_2) \). Applying Lemma 3 with \( a_i = \sum_{\ell \in B} M_{i,\ell}^* \), \( \pi = \tilde{\pi}_{\text{ref}} \) and \( \tau = 2\eta_B \), we see that for all \( i \in [n_1] \),
\[
\left| \sum_{\ell \in B} (M_{\tilde{\pi}_{\text{ref}}(i),\ell}^* - M_{i,\ell}^*) \right| \leq 2\eta_B \leq 96(\zeta + 1) \sqrt{\frac{n_1n_2}{N} |B| \log(n_1n_2)}.
\]
We now analyze the quantities in inequality (27). By the definition of the blocking \( BL \) in the Frobenius error.

Moreover, we have

\[ M \in [0,1]^{n_1 \times n_2} \]

Having established the inequalities (26a) and (26b), we are now tasked with bounding the row-wise approximation error. Critical to our analysis is the following lemma:

**Lemma 7.** For a vector \( v \in \mathbb{R}^n \), define its variation as \( \text{var}(v) = \max_i v_i - \min_i v_i \). Then we have

\[ ||v||^2_2 \leq \text{var}(v)||v||_1 + ||v||^2_1/n. \]

See Section 4.5.1 for the proof of this lemma.

For each \( i \in [n_1] \), define \( \Delta^i \) to be the \( i \)-th row difference \( M^*_\pi(i) - M^*_i \), and for each block \( B \in \text{bl}_{\text{ref}} \), denote the restriction of \( \Delta^i \) to \( B \) by \( \Delta^i_B \). Lemma 7 applied with \( v = \Delta^i_B \) yields

\[ ||\Delta^i||^2_2 = \sum_{B \in \text{bl}_{\text{ref}}} ||\Delta^i_B||^2_2 \]

\[ \leq \sum_{B \in \text{bl}_{\text{ref}}} \text{var}(\Delta^i_B)||\Delta^i_B||_1 + \sum_{B \in \text{bl}_{\text{ref}}} ||\Delta^i_B||^2_1/|B| \]

\[ \leq \left( \max_{B \in \text{bl}_{\text{ref}}} ||\Delta^i_B||_1 \right) \left( \sum_{B \in \text{bl}_{\text{ref}}} \text{var}(\Delta^i_B) \right) + \frac{\max_{B \in \text{bl}_{\text{ref}}} ||\Delta^i_B||_1}{\min_{B \in \text{bl}_{\text{ref}}} |B|} \sum_{B \in \text{bl}_{\text{ref}}} ||\Delta^i_B||_1. \]  

We now analyze the quantities in inequality (27). By the definition of the blocking \( BL \), we have

\[ \frac{1}{2} n_2 \sqrt{\frac{n_1}{N} \log(n_1 n_2)} \leq ||B|| \leq \sqrt{\frac{n_1}{N} \log(n_1 n_2)}. \]

Additionally, the bounds (26a) and (26b) imply that

\[ \sum_{B \in \text{bl}_{\text{ref}}} ||\Delta^i_B||_1 = ||\Delta^i||_1 \leq 96(\zeta + 1)n_2 \sqrt{\frac{n_1}{N} \log(n_1 n_2)}, \]  

and

\[ ||\Delta^i_B||_1 \leq 96(\zeta + 1)n_2 \left( \frac{n_1}{N} \log(n_1 n_2) \right)^{3/4} \]  

for all \( B \in \text{bl}_{\text{ref}} \).

Moreover, we have

\[ \sum_{B \in \text{bl}_{\text{ref}}} \text{var}(\Delta^i_B) \leq \sum_{B \in \text{bl}_{\text{ref}}} \left[ \text{var}(M^*_i,B) + \text{var}(M^*_{\pi(i)},B) \right] \]

\[ \leq \text{var}(M^*_i) + \text{var}(M^*_{\pi(i)}) \leq 2, \]

because \( M^* \) has monotone rows in \([0,1]^{n_2}\). Finally, plugging all the pieces into (27) yields

\[ ||\Delta^i||^2_2 \lesssim (\zeta + 1)n_2 \left( \frac{n_1 \log n_1}{N} \right)^{3/4}. \]

Normalizing this bound by \( 1/n_2 \) completes the proof for the row-wise approximation bound.

Summing the row-wise bounds over the rows and applying Proposition 1, we obtain the bound in the Frobenius error.
4.5.1 Proof of Lemma 7

Let \( a = \min_{i \in [n]} v_i \) and \( b = \max_{i \in [n]} v_i = a + \text{var}(v) \). Since the quantities in the inequality remain the same if we replace \( v \) by \(-v\), we assume without loss of generality that \( b \geq 0 \). If \( a \leq 0 \), then \( \|v\|_\infty \leq b - a = \text{var}(v) \). If \( a > 0 \), then \( a \leq \|v\|_1/n \) and \( \|v\|_\infty = b \leq \|v\|_1/n + \text{var}(v) \). Hence in any case we have \( \|v\|_2^2 \leq \|v\|_\infty \|v\|_1 \leq \|v\|_1/n + \|v\|_1 \).

4.6 Proof of Theorem 4

As mentioned before, an equivalent row-wise approximation bound to equation (9a) was already obtained in previous work [SBW16a, CM16, PMM+17]. In fact, the blocking procedure and comparisons of partial row sums in our algorithm are irrelevant to achieving the bound (9a). Concentration of whole row sums \( S(i) \) suffices to yield the desired rate.

The proof follows immediately from that of Theorem 3. Using the same argument as in that proof leading to the bound (26a), we obtain

\[
\left| \sum_{j \in [n_2]} (M_{i,j}^* - M_{i,j}^*) \right| \leq 96(\zeta + 1) \sqrt{\frac{n_1n_2^2}{N} \log(n_1n_2)} \quad \text{for all } i \in [n_1].
\]

Normalizing this bound by \( 1/n_2 \) completes the proof for the row-approximation error.

For the rest of this section, we focus on showing the Frobenius error bound (9b). The beginning of the proof proceeds in the same way as the proof of Theorem 3, so that we provide only a sketch. We apply Lemma 2 with \( S = \{i\} \times [n_2] \) and \( S = \{i\} \times \mathcal{BL}_k \) for each tuple \( i \in [n_1], k \in [K] \), and use the fact that \( K \leq n_2/\beta \leq n_2^{1/2} \), to obtain that with probability at least \( 1 - 2(n_1n_2)^{-3} \), all the full row sums of \( Y(2) \) and all the partial row sums over the column blocks concentrate well around their means. By virtue of the conditions (8a) and (8b), we see that every edge \( u \rightarrow v \) in the graph \( G \) is consistent with the underlying permutation so that a topological sort exists with probability at least \( 1 - 2(n_1n_2)^{-3} \). Additionally, it follows from Lemma 3 and the same argument leading to equations (26a) and (26b) that for all \( i \in [n_1] \), we have

\[
\left| \sum_{j \in [n_2]} (M_{i,j}^* - M_{i,j}^*) \right| \leq 96(\zeta + 1) \sqrt{\frac{n_1n_2^2}{N} \log(n_1n_2)}, \quad \text{(28a)}
\]

\[
\left| \sum_{j \in \mathcal{BL}_k} (M_{i,j}^* - M_{i,j}^*) \right| \leq 96(\zeta + 1) \sqrt{\frac{n_1n_2^2}{N} |\mathcal{BL}_k| \log(n_1n_2)} \quad \text{for all } k \in [K], \quad \text{(28b)}
\]

with probability at least \( 1 - 2(n_1n_2)^{-3} \).

On the other hand, we apply Lemma 2 with \( S = [n_1] \times \{j\} \) to obtain concentration for the column sums of \( Y(1) \):

\[
\left| C(j) - \sum_{i=1}^{n_1} M_{i,j} \right| \leq 8(\zeta + 1) \left( \sqrt{\frac{n_1^2n_2}{N} \log(n_1n_2)} + 2 \frac{n_1n_2}{N} \log(n_1n_2) \right) \quad \text{(29)}
\]

for all \( j \in [n_2] \) with probability at least \( 1 - 2(n_1n_2)^{-3} \). We carry out the remainder of the proof conditioned on the event of probability at least \( 1 - 4(n_1n_2)^{-3} \) that inequalities (28a), (28b) and (29) hold.
Having stated the necessary bounds, we now split the remainder of the proof into two parts for convenience. In order to do so, we first split the set \( BL \) into two disjoint sets of blocks, depending on whether a block comes from an originally large block (of size larger than \( BL \)) as in Step 3 of Subroutine 1) or from an aggregation of small blocks. More formally, define the sets

\[
BL^L := \{ B \in BL : B \text{ was not obtained via aggregation} \}, \quad \text{and} \\
BL^S := BL \setminus BL^L.
\]

For a set of blocks \( B \), define the shorthand \( \cup B = \bigcup_{B \in B} B \) for convenience. We begin by focusing on the blocks \( BL^L \).

### 4.6.1 Error on columns indexed by \( \cup BL^L \)

Recall that when the columns of the matrix are ordered according to \( \sigma_{pre} \), the blocks in \( BL^L \) are contiguous and thus have an intrinsic ordering. We index the blocks according to this ordering as \( B_1, B_2, \ldots, B_\ell \) where \( \ell = |BL^L| \). Now define the disjoint sets

\[
\begin{align*}
BL^{(1)} &= \{ B_k \in BL^L : k = 0 \pmod{2} \}, \quad \text{and} \\
BL^{(2)} &= \{ B_k \in BL^L : k = 1 \pmod{2} \}.
\end{align*}
\]

Let \( \ell = |BL^{(t)}| \) for each \( t = 1, 2 \).

Recall that each block \( B_k \) in \( BL^L \) remains unchanged after aggregation, and that the threshold we used to block the columns is \( \tau = 16(\zeta + 1)\left( \sqrt{\frac{n_1^2}{N} \log(n_1 n_2) + 2 \frac{n_1 n_2}{N} \log(n_1 n_2)} \right) \). Hence, applying the concentration bound (29) together with the definition of blocks in Step 2 of Subroutine 1 yields

\[
\sum_{i=1}^{n_1} M_{i,j_1} - \sum_{i=1}^{n_2} M_{i,j_2} \leq 96(\zeta + 1)\sqrt{\frac{n_1^2 n_2}{N} \log(n_1 n_2)} \quad \text{for all } j_1, j_2 \in B_k, \tag{31}
\]

where we again used the argument leading to the bounds (26a) and (26b) to combine the two terms. Moreover, since the threshold is twice the concentration bound, it holds that under the true ordering id, every index in \( B_k \) precedes every index in \( B_{k+2} \) for any \( k \in [K-2] \). By definition, we have thus ensured that the blocks in \( BL^{(t)} \) do not “mix” with each other.

The rest of the argument hinges on the following lemma, which is proved in Section 4.6.3.

**Lemma 8.** For \( m \in \mathbb{Z}_+ \), let \( J_1 \sqcup \cdots \sqcup J_\ell \) be a partition of \([m]\) such that each \( J_k \) is contiguous and \( J_k \) precedes \( J_{k+1} \). Let \( a_k = \min J_k \), \( b_k = \max J_k \) and \( m_k = |J_k| \). Let \( A \) be a matrix in \([0,1]^{n \times m}\) with nondecreasing rows and nondecreasing columns. Suppose that

\[
\sum_{i=1}^{n} (A_{i,b_k} - A_{i,a_k}) \leq \tau \quad \text{for each } k \in [\ell] \quad \text{and some } \tau \geq 0.
\]

Additionally, suppose that there are positive reals \( \rho, \rho_1, \rho_2, \ldots, \rho_\ell \), and a permutation \( \pi \) such that for any \( i \in [n] \), we have (i) \( \sum_{j=1}^{m} |A_{\pi(i),j} - A_{i,j}| \leq \rho \), and (ii) \( \sum_{j \in J_k} |A_{\pi(i),j} - A_{i,j}| \leq \rho_k \) for each \( k \in [\ell] \). Then it holds that

\[
\sum_{i=1}^{n} \sum_{j=1}^{m} (A_{\pi(i),j} - A_{i,j})^2 \leq 2\tau \sum_{k=1}^{\ell} \rho_k + n \rho \max_{k \in [\ell]} \frac{\rho_k}{m_k}.
\]

29
We apply the lemma as follows. For $t = 1, 2$, let the matrix $M^{(t)}$ be the submatrix of $M^*$ restricted to the columns indexed by the indices in $\cup BL^{(t)}$. The matrix $M^{(t)}$ has nondecreasing rows and columns by assumption. We have shown that the blocks in $BL^{(t)}$ do not mix with each other, so they are contiguous and correctly ordered in $M^{(t)}$. Moreover, the inequality assumptions of the lemma correspond to (31), (28a) and (28b) respectively, with the substitutions

\[
A = M^{(t)}, \quad n = n_1, \quad m = | \cup BL^{(t)}|, \quad \tau = 96(\zeta + 1) \sqrt[4]{n_1^2 n_2^2 \log(n_1 n_2)}
\]

\[
\rho = 96(\zeta + 1) \sqrt{\frac{n_1^2 n_2^2}{N^2} \log(n_1 n_2)}, \quad \rho_k = 96(\zeta + 1) \sqrt{\frac{n_1^2 n_2^2}{N^2} \log(n_1 n_2)},
\]

and setting $J_1, \ldots, J_\ell$ to be the blocks in $BL^{(t)}$. Therefore, applying Lemma 8 yields

\[
\sum_{i \in [n_1]} \sum_{j \in \cup BL^{(t)}} (M^*_{i,i}-M^*_{i,j})^2 
\]

\[
\leq (\zeta^2 \lor 1) \frac{n_1^{3/2} n_2^2}{N} \log(n_1 n_2) \sum_{B \in BL^{(t)}} \sqrt{|B|} + (\zeta^2 \lor 1) \frac{n_1^{3/2} n_2^2}{N} \log(n_1 n_2) \max_{B \in BL^{(t)}} \frac{\sqrt{|B|}}{|B|}
\]

\[
\leq (\zeta^2 \lor 1) \frac{n_1^{3/2} n_2^2}{N} \log(n_1 n_2) \sum_{B \in BL^{(t)}} |B| \sqrt{\ell_t} + (\zeta^2 \lor 1) \frac{n_1^{3/2} n_2^2}{N} \log(n_1 n_2) \min_{B \in BL^{(t)}} \frac{1}{\sqrt{|B|}}
\]

\[
\leq \frac{(\zeta^2 \lor 1) n_1^{3/2} n_2^2}{\sqrt{\beta}} \log(n_1 n_2) + \frac{(\zeta^2 \lor 1) n_1^{3/2} n_2^2}{\sqrt{\beta}} \log(n_1 n_2)
\]

\[
\leq \frac{(\zeta^2 \lor 1) n_1^{3/2} n_2^2}{\sqrt{\beta}} \log(n_1 n_2) + \frac{(\zeta^2 \lor 1) n_1^{3/2} n_2^2}{\sqrt{\beta}} \log(n_1 n_2)
\]

where step (i) follows from the Cauchy-Schwarz inequality, and step (ii) follows from the fact that $\min_{B \in BL^{(t)}} |B| \geq \beta = n_2 \sqrt{\frac{N}{n_1^2} \log(n_1 n_2)}$ so that $\ell_t = n_2 / \beta$. Substituting for $\beta$ and normalizing by $n_1 n_2$ yields

\[
\frac{1}{n_1 n_2} \sum_{i \in [n_1]} \sum_{j \in \cup BL^{(t)}} (M^*_{i,i}-M^*_{i,j})^2 \leq (\zeta^2 \lor 1) n_1^{1/4} (n_1 \lor n_2)^{1/2} \left( \frac{\log(n_1 n_2)}{N} \right)^{3/4}.
\]

This proves the required result for the set of blocks $BL^{(t)}$. Summing over $t = 1, 2$ then yields a bound of twice the size for columns of the matrix indexed by $\cup BL$.

### 4.6.2 Error on columns indexed by $\cup BL^S$

Next we bound the approximation error of each row of the matrix with column indices restricted to the union of all small blocks. In the easy case where $BL^S$ contains a single block of size less than
\[ \frac{1}{2} n_2 \sqrt{\frac{\log(n_1 n_2)}{N}} \], we have

\[
\sum_{i \in [n_1]} \sum_{j \in \cup BL^S} (M^*_\text{nda}(i), j - M^*_i, j)^2 \leq \sum_{i \in [n_1]} \sum_{j \in \cup BL^S} |M^*_\text{nda}(i), j - M^*_i, j| \\
= \sum_{i \in [n_1]} |\sum_{j \in \cup BL^S} (M^*_\text{nda}(i), j - M^*_i, j)| \\
\leq \sum_{i \in [n_1]} 96(\zeta + 1) \sqrt{\frac{n_1 n_2}{2N} \left[ \frac{n_1}{N} \right]^{1/2} \log^{3/2}(n_1 n_2)} \\
= 48 \sqrt{2(\zeta + 1)} \frac{n_1^{3/2} n_2 (n_1 \vee n_2)^{1/2}}{N^{3/4}} \log^{3/4}(n_1 n_2),
\]

where step (i) follows from the Hölder’s inequality and the fact that \( M^* \in [0, 1]^{n_1 \times n_2} \), step (ii) from the monotonicity of the columns of \( M^* \), and step (iii) from equation (28a).

For each \( i \in [n_1] \), define \( \Delta^i \) to be the restriction of the \( i \)-th row difference \( M^*_\text{nda}(i) - M^*_i \) to the union of blocks \( \cup BL^S \). For each block \( B \in BL^S \), denote the restriction of \( \Delta^i \) to \( B \) by \( \Delta^i_B \). Lemma 7 applied with \( v = \Delta^i \) yields

\[
\|\Delta^i\|^2_2 = \sum_{B \in BL^S} \|\Delta^i_B\|^2_2 \\
\leq \sum_{B \in BL^S} \operatorname{var}(\Delta^i_B) \|\Delta^i_B\|_1 + \sum_{B \in BL^S} \frac{\|\Delta^i_B\|_1}{|B|} \\
\leq \left( \max_{B \in BL^S} \|\Delta^i_B\|_1 \right) \sum_{B \in BL^S} \operatorname{var}(\Delta^i_B) + \frac{\max_{B \in BL^S} \|\Delta^i_B\|_1}{\min_{B \in BL^S} |B|} \sum_{B \in BL^S} \|\Delta^i_B\|_1 \\
\leq \left( \max_{B \in BL^S} \|\Delta^i_B\|_1 \right) \left( \sum_{B \in BL^S} \operatorname{var}(\Delta^i_B) \right) + \frac{\max_{B \in BL^S} \|\Delta^i_B\|_1}{\min_{B \in BL^S} |B|} \sum_{B \in BL^S} \|\Delta^i_B\|_1.
\]

We now analyze the quantities in inequality (33). By the aggregation step of Subroutine 1, we have \( \frac{1}{2} \beta \leq |B| \leq 2 \beta \), where \( \beta = n_2 \sqrt{\frac{\log(n_1 n_2)}{N}} \). Additionally, the bounds (28a) and (28b) imply that

\[
\sum_{B \in BL^S} \|\Delta^i_B\|_1 = \|\Delta^i\|_1 \leq 96(\zeta + 1) \sqrt{\frac{n_1 n_2}{2N}} \log(n_1 n_2) \lesssim (\zeta + 1) \beta, \quad \text{and}
\]

\[
\|\Delta^i_B\|_2 \leq 96(\zeta + 1) \sqrt{\frac{n_1 n_2}{N}} |B| \log(n_1 n_2) \\
\leq 96 \sqrt{2(\zeta + 1)} \sqrt{\frac{n_1 n_2}{N}} \beta \log(n_1 n_2) \quad \text{for all } B \in BL^S.
\]

Moreover, to bound the quantity \( \sum_{B \in BL^S} \operatorname{var}(\Delta^i_B) \), we proceed as in the proof for the large blocks in \( BL^L \). Recall that if we permute the columns by \( \bar{\sigma}_\text{pre} \) according to the column sums, then the blocks in \( BL^S \) have an intrinsic ordering, even after adjacent small blocks are aggregated. Let us index the blocks in \( BL^S \) by \( B_1, B_2, \ldots, B_m \) according to this ordering, where \( m = |BL^S| \). As before,
the odd-indexed (or even-indexed) blocks do not mix with each other under the true ordering \( \pi \), because the threshold used to define the blocks is larger than twice the column sum perturbation. We thus have

\[
\sum_{B \in BL^S} \text{var}(\Delta^i) = \sum_{k \in [m] \atop k \text{ odd}} \text{var}(\Delta^i_{B_k}) + \sum_{k \in [m] \atop k \text{ even}} \text{var}(\Delta^i_{B_k}) \\
\leq \sum_{k \in [m] \atop k \text{ odd}} \left[ \text{var}(M^*_{i,B_k}) + \text{var}(M^*_{\pi_{\text{lds}}(i),B_k}) \right] + \sum_{k \in [m] \atop k \text{ even}} \left[ \text{var}(M^*_{i,B_k}) + \text{var}(M^*_{\pi_{\text{lds}}(i),B_k}) \right] \\
\overset{(i)}{\leq} 2 \text{var}(M^*_i) + 2 \text{var}(M^*_{\pi_{\text{lds}}(i)}) \leq 4,
\]

where inequality (i) holds because the odd (or even) blocks do not mix, and inequality (ii) holds because \( M^* \) has monotone rows in \([0, 1]^{n_2}\).

Finally, putting together all the pieces, we can substitute for \( \beta \), sum over the indices \( i \in n_1 \), and normalize by \( n_1 n_2 \) to obtain

\[
\frac{1}{n_1 n_2} \sum_{i \in [n_1]} \|\Delta^i\|^2 \lesssim (\epsilon^2 \lor 1) \left( \frac{n_1 \log(n_1 n_2)}{N} \right)^{3/4},
\]

and so the error on columns indexed by the set \( \cup BL^S \) is bounded as desired.

Combining the bounds (32) and (34), we conclude that

\[
\frac{1}{n_1 n_2} \|M^*(\pi_{\text{lds}}, \text{id}) - M^*\|_F^2 \lesssim (\epsilon^2 \lor 1) n_1^{1/4} (n_1 \lor n_2)^{1/2} \left( \frac{\log(n_1 n_2)}{N} \right)^{3/4}
\]

with probability at least \( 1 - 4(n_1 n_2)^{-3} \). The same proof works with the roles of \( n_1 \) and \( n_2 \) switched and all the matrices transposed, so it holds with the same probability that

\[
\frac{1}{n_1 n_2} \|M^*(\text{id}, \sigma_{\text{lds}}) - M^*\|_F^2 \lesssim (\epsilon^2 \lor 1) n_2^{1/4} (n_1 \lor n_2)^{1/2} \left( \frac{\log(n_1 n_2)}{N} \right)^{3/4}.
\]

Consequently,

\[
\frac{1}{n_1 n_2} \left( \|M^*(\pi_{\text{lds}}, \text{id}) - M^*\|_F^2 + \|M^*(\text{id}, \sigma_{\text{lds}}) - M^*\|_F^2 \right) \lesssim (\epsilon^2 \lor 1) \left( \frac{n_1 \log n_1}{N} \right)^{3/4}
\]

with probability at least \( 1 - 8(n_1 n_2)^{-3} \), where we have used the relation \( n_1 \geq n_2 \). Applying Proposition 1 completes the proof.

### 4.6.3 Proof of Lemma 8

Since \( A \) has increasing rows, for any \( i, i_2 \in [n] \) with \( i \leq i_2 \) and any \( j, j_2 \in J_k \), we have

\[
A_{i_2,j} - A_{i,j} = (A_{i_2,j} - A_{i_2,a_k}) + (A_{i_2,a_k} - A_{i,b_k}) + (A_{i,b_k} - A_{i,j}) \\
\leq (A_{i_2,b_k} - A_{i_2,a_k}) + (A_{i_2,j_2} - A_{i,j_2}) + (A_{i,b_k} - A_{i,a_k}).
\]
Choosing $j_2 = \arg\min_{r \in J_k} (A_{i_2,r} - A_{i,r})$, we obtain

$$A_{i_2,j} - A_{i,j} \leq (A_{i_2,b_k} - A_{i_2,a_k}) + (A_{i,b_k} - A_{i,a_k}) + \frac{1}{m_k} \sum_{r \in J_k} (A_{i_2,r} - A_{i,r}).$$

Together with the assumption on $\pi$, this implies that

$$|A_{\pi(i),j} - A_{i,j}| \leq \underbrace{A_{\pi(i),b_k} - A_{\pi(i),a_k}}_{\chi_{i,k}} + \underbrace{A_{i,b_k} - A_{i,a_k}}_{\eta_{i,k}} + \frac{1}{m_k} \sum_{r \in J_k} |A_{\pi(i),r} - A_{i,r}|.$$

Hence it follows that

$$\sum_{i=1}^{n} \sum_{j=1}^{m} (A_{i,j} - A_{\pi(i),j})^2 = \sum_{i=1}^{n} \sum_{k=1}^{\ell} \sum_{j \in J_k} (A_{i,j} - A_{\pi(i),j})^2$$

$$\leq \sum_{i=1}^{n} \sum_{k=1}^{\ell} \sum_{j \in J_k} |A_{i,j} - A_{\pi(i),j}|(x_{i,k} + y_{i,k} + z_{i,k}/m_k)$$

$$= \sum_{i=1}^{n} \sum_{k=1}^{\ell} z_{i,k}(x_{i,k} + y_{i,k} + z_{i,k}/m_k).$$

According to the assumptions, we have

1. $\sum_{k=1}^{\ell} x_{i,k} \leq 1$ and $\sum_{i=1}^{n} x_{i,k} \leq \tau$ for any $i \in [n], k \in [\ell]$;
2. $\sum_{k=1}^{\ell} y_{i,k} \leq 1$ and $\sum_{i=1}^{n} y_{i,k} \leq \tau$ for any $i \in [n], k \in [\ell]$;
3. $z_{i,k} \leq \rho_k$ and $\sum_{k=1}^{\ell} z_{i,k} \leq \rho$ for any $i \in [n], k \in [\ell]$.

Consequently, the following bounds hold:

1. $\sum_{i=1}^{n} \sum_{k=1}^{\ell} z_{i,k}^2 x_{i,k} \leq \sum_{i=1}^{n} \sum_{k=1}^{\ell} \rho_k x_{i,k} \leq \tau \sum_{k=1}^{\ell} \rho_k$;
2. $\sum_{i=1}^{n} \sum_{k=1}^{\ell} z_{i,k} y_{i,k} \leq \sum_{i=1}^{n} \sum_{k=1}^{\ell} \rho_k y_{i,k} \leq \tau \sum_{k=1}^{\ell} \rho_k$;
3. $\sum_{i=1}^{n} \sum_{k=1}^{\ell} z_{i,k}^2 / m_k \leq \sum_{i=1}^{n} \sum_{k=1}^{\ell} z_{i,k} \cdot \max_{k \in [\ell]} (\rho_k/m_k) \leq n \rho \max_{k \in [\ell]} (\rho_k/m_k)$.

Combining these inequalities yields the claim.

## 5 Discussion

While we have characterized the minimax rate of estimation in the row-wise approximation metric, and narrowed the statistical-computational gap for estimation in Frobenius norm, several intriguing questions related to estimating such matrices remain:

- What is the fastest Frobenius error rate achievable by computationally efficient estimators?
- Can the techniques from here be used to narrow statistical-computational gaps in other permutation-based models [SBW16b, FMR16, PWC17]?
As a partial answer to the first question, it can be shown that when the informal algorithm described at the beginning of Section 3.2.2 is recursed in the natural way and applied to the noisy sorting subclass of the SST model, it yields another minimax optimal estimator for noisy sorting, similar to the multistage algorithm of Mao et al. [MWR17]. However, this same guarantee is preserved for neither the larger class of matrices $C_r Perm$, nor for its sub-class $C_r Perm$. Improving the rate will likely require techniques that are beyond the reach of those introduced in this paper.

It is also worth noting that the model (1) allowed us to perform multiple sample-splitting steps while preserving the independence across observations. While our proofs also hold for the observation model where we have exactly 3 independent samples per entry of the matrix, handling the weak dependence of the original sampling model [Cha15, SBGW17] with exactly one noisy observation per entry is an interesting technical challenge that may also involve its own statistical-computational tradeoffs [Mon15].

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A Additional proofs and helper lemmas

In this section, we collect deferred proofs from the main text, and state and prove auxiliary lemmas.

A.1 Proof of Lemma 1

The proof parallels that of Shah et al. [SBGW17, Theorem 5(a)], so we only emphasize the differences and sketch the remaining argument. We may assume that $p_{obs} \geq \frac{1}{n^2}$, since otherwise the bound is trivial.

We first employ a truncation argument. Consider the event

$$\mathcal{E} := \left\{ |W_{i,j}| \leq \frac{c_3}{p_{obs}} (\zeta \vee 1) \sqrt{\log(n_1 n_2)} \text{ for all } i \in [n_1], j \in [n_2] \right\}.$$

If the universal constant $c_3$ is chosen to be sufficiently large, then it follows from the sub-Gaussianity of $W_{i,j}$ and a union bound over all index pairs $(i, j) \in [n_1] \times [n_2]$ that $\Pr(\mathcal{E}) \geq 1 - (n_1 n_2)^{-4}$. Now define the truncation operator

$$T_\lambda(x) := \begin{cases} x & \text{if } |x| \leq \lambda, \\ \lambda \cdot \text{sgn}(x) & \text{otherwise.} \end{cases}$$

With the choice $\lambda = \frac{c_3}{p_{obs}} (\zeta \vee 1) \sqrt{\log(n_1 n_2)}$, define the random variables $W^{(1)}_{i,j} = T_\lambda(W_{i,j})$ for each pair of indices $(i, j) \in [n_1] \times [n_2]$. Consider the model where we observe $M^* + W^{(1)}$ instead of $Y = M^* + W$. Then the new model and the original one are coupled so that they coincide on the event $\mathcal{E}$. Therefore, it suffices to prove a high probability bound assuming that the noise is given by $W^{(1)}$. 

34
Let us define $\mu = \mathbb{E}[W^{(1)}]$ and $\widetilde{W} = W^{(1)} - \mu$. We claim that for any $i \in [n_1], j \in [n_2]$, the following relations hold:

1. $|\mu_{i,j}| \leq \frac{c}{p_{\text{obs}}} (\zeta \lor 1)(n_1 n_2)^{-4}$;
2. $\widetilde{W}_{i,j}$ are independent, centered and $\frac{c}{p_{\text{obs}}} (\zeta \lor 1)$-sub-Gaussian;
3. $|\widetilde{W}_{i,j}| \leq \frac{c}{p_{\text{obs}}} (\zeta \lor 1) \sqrt{\log(n_1 n_2)}$;
4. $\mathbb{E}[|\widetilde{W}_{i,j}|^2] \leq \frac{c}{p_{\text{obs}}} (\zeta^2 \lor 1)$.

Taking these claims as given for the moment, we turn to the main argument assuming that our observations take the form $Y = \mathcal{M}^* + \widetilde{W} + \mu$.

For any permutations $\pi \in S_{n_1}, \sigma \in S_{n_2}$, let $\mathcal{M}_{\pi,\sigma} = \widehat{M}_{\text{LS}}(Y, C^{\iota}_{\text{Perm}}(\pi, \sigma))$. We claim that for any fixed pair $(\pi, \sigma)$ such that $\|Y - \mathcal{M}_{\pi,\sigma}\|^2_F \leq \|Y - \mathcal{M}^*\|^2_F$, we have

$$\Pr\left\{ \|\mathcal{M}_{\pi,\sigma} - \mathcal{M}^*\|^2_F \geq c_1 (\zeta^2 \lor 1) \frac{n_1}{p_{\text{obs}}} \log^2 n_1 \right\} \leq n_1^{-3n_1}. \quad (36)$$

Treating claim (36) as true for the moment, we see that since the least squares estimator $\widehat{M}$ is equal to $\mathcal{M}_{\pi,\sigma}$ for some pair $(\pi, \sigma)$, a union bound over $\pi \in S_{n_1}, \sigma \in S_{n_2}$ yields

$$\Pr\left\{ \|\widehat{M} - \mathcal{M}^*\|^2_F \geq c_1 (\zeta^2 \lor 1) \frac{n_1}{p_{\text{obs}}} \log^2 n_1 \right\} \leq n_1^{-n_1},$$

which completes the proof. Thus, to prove our result, it suffices to prove claim (36).

Let $\Delta_{\pi,\sigma} = \mathcal{M}_{\pi,\sigma} - \mathcal{M}^*$. The condition $\|Y - \mathcal{M}_{\pi,\sigma}\|^2_F \leq \|Y - \mathcal{M}^*\|^2_F$ yields the basic inequality

$$\frac{1}{2} \|\Delta_{\pi,\sigma}\|^2_F \leq \langle \Delta_{\pi,\sigma}, \widetilde{W} + \mu \rangle.$$

Since $\Delta_{\pi,\sigma} \in [-1, 1]^{n_1 \times n_2}$, we have $\langle \Delta_{\pi,\sigma}, \mu \rangle \leq \|\mu\|_1 \leq \frac{c}{p_{\text{obs}}} (\zeta \lor 1) n_1^{-6}$ by claim 1. If it holds that $\|\Delta_{\pi,\sigma}\|^2_F \leq \frac{4c}{p_{\text{obs}}} (\zeta \lor 1) n_1^{-6}$, then the proof is immediate. Thus, we may assume the opposite, from which it follows that

$$\frac{1}{4} \|\Delta_{\pi,\sigma}\|^2_F \leq \langle \Delta_{\pi,\sigma}, \widetilde{W} \rangle. \quad (37)$$

Consider the set of matrices

$$\mathcal{C}_{\text{DIFF}}(\pi, \sigma) := \{ \alpha (M - \mathcal{M}^*) : M \in \mathcal{C}_{\text{BISO}}(\pi, \sigma), \alpha \in [0, 1] \}.$$

Additionally, for every $t > 0$, define the random variable

$$Z_{\pi,\sigma}(t) := \sup_{D \in \mathcal{C}_{\text{DIFF}}(\pi, \sigma), \|D\|_F \leq t} \langle D, \widetilde{W} \rangle.$$

For every $t > 0$, define the event

$$\mathcal{A}_t := \left\{ \text{there exists } D \in \mathcal{C}_{\text{DIFF}}(\pi, \sigma) \text{ such that } \|D\|_F \geq \sqrt{t \delta_n} \text{ and } \langle D, \widetilde{W} \rangle \geq 4\|D\|_F \sqrt{t \delta_n} \right\}.$$
For $t \geq \delta_n$, either we already have $\|\Delta_{\pi,\sigma}\|_F^2 \leq t\delta_n$, or we have $\|\Delta_{\pi,\sigma}\|_F > \sqrt{t\delta_n}$. In the latter case, on the complement of $\mathcal{A}_t$, we must have $\langle \Delta_{\pi,\sigma}, \tilde{W} \rangle \leq 4\|\Delta_{\pi,\sigma}\|_F \sqrt{t\delta_n}$. Combining this with inequality (37) then yields $\|\Delta_{\pi,\sigma}\|_F^2 \leq c_5\delta_n$. It thus remains to bound the probability $\Pr\{\mathcal{A}_t\}$.

Using the star-shaped nature of the set $\mathcal{C}_{\text{DIFF}}(\pi, \sigma)$, a rescaling argument yields

$$\Pr\{\mathcal{A}_t\} \leq \Pr\left\{ Z_{\pi,\sigma}(\delta_n) \geq 4\delta_n \sqrt{t\delta_n} \right\} \quad \text{for all } t \geq \delta_n.$$ 

The following lemma bounds the tail behavior of the random variable $Z_{\pi,\sigma}(\delta_n)$, and its proof is postponed to Section A.1.2.

**Lemma 9.** For any $\delta > 0$ and $u > 0$, we have

$$\Pr\left\{ Z_{\pi,\sigma}(\delta) > \frac{c}{P_{\text{obs}}} (\zeta \vee 1) \sqrt{\log n_1} (n_1 \log^{1.5} n + u) \right\} \leq \exp\left( \frac{-c_1 u^2}{P_{\text{obs}} \delta^2 / (\log n_1) + n_1 \log^{1.5} n_1 + u} \right).$$

Taking the lemma as given and setting $\delta_n^2 = \frac{c_2}{P_{\text{obs}}} (\zeta^2 \vee 1) n_1 \log^2 n_1$ and $u = c_3 (\zeta \vee 1) n_1 \log^{1.5} n_1$, we see that for any $t \geq \delta_n$, we have

$$\Pr\{\mathcal{A}_t\} \leq \Pr\left\{ Z_{\pi,\sigma}(\delta_n) \geq 4\delta_n \sqrt{t\delta_n} \right\} \leq \exp\left( \frac{-c_4 (\zeta^2 \vee 1) n_1^2 \log^3 n_1}{(\zeta^2 \vee 1) n_1 \log n_1 + n_1 \log^{1.5} n_1} \right) \leq n_1^{-3n_2}. \quad (38)$$

In particular, for $t = \delta_n$, on the complement of $\mathcal{A}_t$, we have

$$\|\Delta_{\pi,\sigma}\|_F^2 \leq \frac{c_5}{P_{\text{obs}}} (\zeta^2 \vee 1) n_1 \log^2 n_1,$$

which completes the proof. Note that the original proof sacrificed a logarithmic factor in proving the equivalent of equation (38), and this is why we recover the same logarithmic factors as in the bounded case in spite of the sub-Gaussian truncation argument.

In the setting where we know that $M^* \in \mathcal{C}_{\text{BISO}}$, the same proof clearly works, except that we do not even need to take a union bound over $\pi \in \mathcal{S}_{n_1}, \sigma \in \mathcal{S}_{n_2}$ as the columns and rows are ordered.

### A.1.1 Proof of claims 1–4

We assume throughout that the constant $c_3$ is chosen to be sufficiently large. Claim 1 follows as a result of the following argument; we have

$$|\mu_{i,j}| = \left| \mathbb{E}[W_{i,j}^{(1)}] \right| 
\leq \mathbb{E}\left[ |W_{i,j}^{(1)} - W_{i,j}| \right] 
= \int_0^\infty \Pr\{|W_{i,j}^{(1)} - W_{i,j}| \geq t\} dt 
= \int_0^\infty \Pr\{|W_{i,j}| \geq \frac{c_3}{P_{\text{obs}}} (\zeta \vee 1) \sqrt{\log(n_1n_2)} + t\} dt 
\leq (n_1n_2)^{-5} \int_0^\infty \exp\left( \frac{-t^2}{c_4 (\zeta^2 \vee 1) P_{\text{obs}}^2} \right) dt 
\leq \frac{c_5}{P_{\text{obs}}} (\zeta \vee 1)(n_1n_2)^{-4}.$$

36
By definition, the random variables $W_{i,j}^{(1)} - \mu_{i,j}$ are independent and zero-mean, and applying Lemma 12 (see Appendix A.4) yields that they are also sub-Gaussian with the claimed variance parameter, thus yielding claim 2. The triangle inequality together with the definition of $\tilde{W}_{i,j}$ then yields claim 3.

Finally, since $|T(x)| \leq |x|$, we have

\[
E[|\tilde{W}_{i,j}|^2] \leq E[|W_{i,j}^{(1)}|^2] \leq E[|W_{i,j}|^2] \leq \frac{c_6}{p_{\text{obs}}} (\zeta^2 \lor 1),
\]

yielding claim 4.

A.1.2 Proof of Lemma 9

The chaining argument from the proof of Shah et al. [SBGW17, Lemma 10] can applied to show that

\[
E[Z_{\pi,\sigma}(\delta)] \leq \frac{c_2}{p_{\text{obs}}} (\zeta \lor 1) n_1 \log^2 n_1,
\]

as $\tilde{W}_{i,j}$ is $\frac{c}{p_{\text{obs}}} (\zeta \lor 1)$-sub-Gaussian by claim 2. Note that although we are considering a set of rectangular matrices $C_{\text{DIFF}}(\pi, \sigma) \subset [-1,1]^{n_1 \times n_2}$ instead of square matrices as in [SBGW17], we can augment each matrix by zeros to obtain an $n_1 \times n_1$ matrix, and so $C_{\text{DIFF}}(\pi, \sigma)$ can be viewed as a subset of its counterpart consisting of $n_1 \times n_1$ matrices. Hence the entropy bound depending on $n_1$ can be employed so that the chaining argument indeed goes through.

In order to obtain the deviation bound, we apply Lemma 11 of Shah et al. [SBGW17] (i.e., Theorem 1.1(c) of Klein and Rio [KR05]) with $V = C_{\text{DIFF}}(\pi, \sigma) \cap B_\delta$, $m = n_1 n_2$, $X = \frac{p_{\text{obs}}}{c(\zeta \lor 1) \sqrt{\log n_1}} \tilde{W}$ and $X^\dagger = \frac{p_{\text{obs}}}{c(\zeta \lor 1) \sqrt{\log n_1}} Z_{\pi,\sigma}(\delta)$. Claim 3 guarantees that $|X|$ is uniformly bounded by 1. We also have $E[\langle D, \tilde{W} \rangle^2] \leq \frac{c}{p_{\text{obs}}} (\zeta^2 \lor 1) \delta^2$ by claim 4 for $\|D\|_F^2 \leq \delta^2$. Therefore, we conclude that

\[
\Pr \left\{ Z_{\pi,\sigma}(\delta) > E[Z_{\pi,\sigma}(\delta)] + \frac{c}{p_{\text{obs}}} (\zeta \lor 1) \sqrt{\log n_1} \cdot u \right\} \leq \exp \left( \frac{-c_1 u^2}{p_{\text{obs}} \delta^2 \left( \log n_1 + n_1 \log^{1.5} n_1 + u \right)} \right).
\]

Combining the expectation and the deviation bounds completes the proof.

A.2 Proof of Theorem 2 for Bernoulli observations

Most of the proof for Gaussian observations remains valid, so we only discuss the differences. Analogous to definition (13) in the case of Gaussian noise, define the distribution $F(x)$ of observations so that its conditional distribution on an instance of $\kappa$ is

\[
F(x; \kappa) := \otimes_{i=1}^{n_2} \otimes_{\ell=1}^{k_i} \text{Ber}(x_i).
\]

Note that the only ingredient of the proof of Theorem 2 that uses Gaussianity is Lemma 6, so it suffices to replace it with the following lemma.

Lemma 10. Let $\{\theta^0, \theta^1, \ldots, \theta^s\}$ denote $s+1$ vectors in $[1/4, 3/4]^d$. For some two constants $1/4 \leq a \leq b \leq 3/4$, assume that for every $i \in [d]$, we have either

(i) $\theta_i^s - 1 = \theta_i^t$, or
(ii) \( \theta_i^s = a \) and \( \theta_i^{s-1} = b \).

Choose indices \( \alpha \) and \( \beta \) uniformly at random from the sets \([s]\) and \( \{0, 1, \ldots, s - 1\} \) respectively. With the choices \( P = F(\theta^s) \) and \( Q = F(\theta^3) \), we have

\[
\text{KL}(P, Q) \leq s^{-1} \left( 2.1 + 4\lambda\|\theta^s - \theta^{s-1}\|_2^2 + 4\sqrt{\lambda}\|\theta^s - \theta^{s-1}\|_2 \right).
\]

Although neither condition (i) nor (ii) was assumed in Lemma 6, the vectors we constructed for the mixture distributions in fact satisfy either condition (i) or (ii), so Lemma 10 can be applied. Moreover, the constants in the bound are slightly worse than those in Lemma 6, but we can easily adjust the constants in the rest of the proof accordingly. Therefore, it remains to prove Lemma 10.

**A.2.1 Proof of Lemma 10**

The proof is structurally similar to that of Lemma 6, so we adopt the same notation and focus on the differences. We again condition on each instance of \( \kappa \) and obtain

\[
\text{KL}(P, Q) = \mathbb{E}_{\kappa} \mathbb{E}_P \left[ \log \frac{dP}{dQ} \mid \kappa \right].
\]

For each \( j \in \{0\} \cup [s] \), let \( p^j \) be the density\(^3\) of \( F(\theta^j; \kappa) \). In other words, if \( \mu^j \) denotes the vector in \( \mathbb{R}^{\|\kappa\|_1} \) that is the Cartesian product of \( \kappa_i \) copies of \( \theta^j_i \) over all \( i \in [d] \), then \( p^j \) is the density of the distribution having independent Bernoulli entries with mean \( \mu^j \). Therefore, we can write

\[
p^j(y) = \prod_{i=1}^{d} \prod_{\ell=1}^{\kappa_i} f(y_{i,\ell}; \theta^j_i)
\]

(40)

where \( f(\cdot; c) \) denotes the density of \( \text{Ber}(c) \).

Proceeding as in the proof of Lemma 6 up to inequality (19), we can again derive the bound

\[
\mathbb{E}_P \left[ \log \frac{dP}{dQ} \mid \kappa \right] \leq 2.1s^{-1} + 2s^{-1} \int_{p^s_{\ell > 2}} p^s \log \frac{p^s}{p^{s-1}}.
\]

(41)

Different from the Gaussian case where we could exploit rotational invariance, in the Bernoulli case we need to study the above integral by explicitly analyzing the marginals of \( p^j \). Towards this end, let us denote by \( \tilde{p}^j_{i,\ell} \) the density of \( p^j \) marginalized over the coordinate \( y_{i,\ell} \). Equivalently, we have

\[
\tilde{p}^j_{i,\ell}(\tilde{y}_{i,\ell}) = p^j(y) / f(y_{i,\ell}; \theta^j_i),
\]

where \( \tilde{y}_{i,\ell} \) is simply the variable \( y \) without the coordinate \( y_{i,\ell} \).

First, we apply equation (40) to obtain

\[
\int_{p^s_{\ell > 2}} p^s \log \frac{p^s}{p^{s-1}} = \sum_{i=1}^{d} \sum_{\ell=1}^{\kappa_i} \int_{p^s_{\ell > 2}} p^s(y) \log \frac{f(y_{i,\ell}; \theta^j_i)}{f(y_{i,\ell}; \theta^{s-1}_i)}.
\]

Now we explicitly calculate the integral over the \((i, \ell)\)-th coordinate, which is simply a sum of two terms depending on the binary value of \( y_{i,\ell} \). Since \( y_{i,\ell} = 1 \) with probability \( \theta^j_i \) and \( y_{i,\ell} = 0 \) with

\[^3\text{We refer to probability mass functions as “densities” for the rest of the proof.}\]
probability $1 - \theta_i^s$ under the density $p^j$, it holds

$$
\int_{p^s > 2} p^s \log \frac{p^s}{p^{s-1}}
= \sum_{i=1}^{d} \sum_{\ell=1}^{\kappa_i} \left( \int_{\frac{\rho_{i,\ell}^s}{\rho_{i,\ell}^{s-1}} > 2} \tilde{p}_{i,\ell}^s \cdot \theta_i^s \log \frac{\theta_i^s}{\theta_i^{s-1}} + \int_{\frac{\rho_{i,\ell}^s}{\rho_{i,\ell}^{s-1}} \leq 2} \tilde{p}_{i,\ell}^s (1 - \theta_i^s) \log \frac{1 - \theta_i^s}{1 - \theta_i^{s-1}} \right),}
$$

(42)

By assumption, we have that either $\theta_i^s = \theta_i^{s-1}$, or $\theta_i^s = a < b = \theta_i^{s-1}$. The terms of (42) where $\theta_i^s = \theta_i^{s-1}$ all vanish as $\log 1 = 0$. It therefore suffices to study the terms where $\theta_i^s = a < b = \theta_i^{s-1}$. Assume that there are a total of $T + 1$ such $(i, \ell)$ pairs; each summand in equation (42) is equal to

$$
\int_{\frac{\rho_{i,\ell}^s}{\rho_{i,\ell}^{s-1}} > 2} \tilde{p}_{i,\ell}^s \cdot a \log \frac{a}{b} + \int_{\frac{\rho_{i,\ell}^s}{\rho_{i,\ell}^{s-1}} \leq \frac{1}{a}} \tilde{p}_{i,\ell}^s \cdot (1 - a) \log \frac{1 - a}{1 - b}
= - \int_{\frac{\rho_{i,\ell}^s}{\rho_{i,\ell}^{s-1}} < \frac{1}{a}} \tilde{p}_{i,\ell}^s \cdot a \log \frac{a}{b} + \int_{\frac{\rho_{i,\ell}^s}{\rho_{i,\ell}^{s-1}} > \frac{1}{a}} \tilde{p}_{i,\ell}^s \cdot \left( a \log \frac{a}{b} + (1 - a) \log \frac{1 - a}{1 - b} \right)
\leq \int_{\frac{\rho_{i,\ell}^s}{\rho_{i,\ell}^{s-1}} < \frac{1}{a}} \tilde{p}_{i,\ell}^s \cdot a \log \frac{b}{a} + \text{KL}(\text{Ber}(a), \text{Ber}(b)),
$$

(43)

where the inequality follows from the definition of the KL divergence and bounding the second integral by 1. The KL divergence satisfies the standard bound

$$
\text{KL}(\text{Ber}(a), \text{Ber}(b)) = a \log \frac{a}{b} + (1 - a) \log \frac{1 - a}{1 - b} = \int_a^b \left( \frac{1 - a - x}{1 - x} \right) dx
= \int_a^b \frac{x - a}{x(1 - x)} dx \leq 4 \int_a^b (x - a) dx = 2(a - b)^2,
$$

(44)

where we used that $x \in [a, b] \subset [1/4, 3/4]$ for the inequality.

We now claim that

$$
\int_{\frac{\rho_{i,\ell}^s}{\rho_{i,\ell}^{s-1}} < \frac{1}{a}} \tilde{p}_{i,\ell}^s \leq \frac{1}{\sqrt{T}}
$$

(45)

Taking the claim as given for the moment and using the inequality $a \log \frac{b}{a} \leq b - a$ which holds for $a$ and $b$ in the given range, we have

$$
\int_{p^s > 2} p^s \log \frac{p^s}{p^{s-1}} \leq (T + 1) \left( \frac{b - a}{\sqrt{T}} + 2(a - b)^2 \right)
\leq 2\sqrt{T} + 1 (b - a) + 2(T + 1)(a - b)^2
\overset{(i)}{=} \|\mu^s - \mu^{s-1}\|_2 + 2\|\mu^s - \mu^{s-1}\|_2^2,
$$

where step (i) holds because $\mu^j$ is the Cartesian product of $\kappa_i$ copies of $\theta_i^j$ over all $i \in [d]$. 

39
This bound plays the same role as the bound (22) in the Gaussian case, and the rest of the proof carries over with constants modified accordingly. That is, it suffices to substitute the last bound into (41), then back into (39) and take the expectation over $\kappa$. The proof is thus complete.

It remains to prove claim (45). Note that the integral on the LHS is the probability of the event \( \{2 \frac{1-b}{a} < \tilde{p}_{i,\ell}/\tilde{p}_{i,\ell}^{-1} \leq 2 \frac{b}{a}\} \) under the density $\tilde{p}_{i,\ell}$. Recall that there are $T+1$ coordinates $j$ such that $\mu_j = a < b = \mu_j^{-1}$. Under the density $\tilde{p}_{i,\ell}$, the observations on the $T$ coordinates (excluding the coordinate corresponding to the pair $(i, \ell)$) are independent $\text{Ber}(a)$ random variables. Therefore, with probability $\binom{T}{k} a^k (1-a)^{T-k}$, we observe $k$ ones on the $T$ coordinates, so that

\[
\frac{\tilde{p}_{i,\ell}}{\tilde{p}_{i,\ell}^{-1}} = \frac{a^k (1-a)^{T-k}}{b^k (1-b)^{T-k}},
\]

for any $k \in \{0\} \cup \{T\}$. Substituting equation (46) into the constraint $2 \frac{1-b}{a} < \frac{\tilde{p}_{i,\ell}}{\tilde{p}_{i,\ell}^{-1}} \leq 2 \frac{b}{a}$, straightforward computation yields

\[
\frac{T \log \frac{1-a}{1-b} - \log \frac{2b}{a}}{\log \frac{b(1-a)}{a(1-b)}} \leq k < \frac{T \log \frac{1-a}{1-b} - \log \frac{2b}{1-a}}{\log \frac{b(1-a)}{a(1-b)}}.
\]

Therefore, the number of such integers $k$ is at most

\[
\frac{T \log \frac{1-a}{1-b} - \log \frac{2b}{1-a}}{\log \frac{b(1-a)}{a(1-b)}} - \frac{T \log \frac{1-a}{1-b} - \log \frac{2b}{a}}{\log \frac{b(1-a)}{a(1-b)}} = \frac{\log \frac{2b}{a} - \log \frac{2b}{1-a}}{\log \frac{b(1-a)}{a(1-b)}} = \frac{\log \frac{b(1-a)}{a(1-b)}}{\log \frac{b(1-a)}{a(1-b)}} = 1.
\]

Consequently, the probability of the event $\{2 \frac{1-b}{a} < \tilde{p}_{i,\ell}/\tilde{p}_{i,\ell}^{-1} \leq 2 \frac{b}{a}\}$ under the density $\tilde{p}_{i,\ell}$ is at most

\[
\max_{0 \leq k \leq T} \binom{T}{k} a^k (1-a)^{T-k}.
\]

This objective is simply the Binomial density, and it is well known that the largest value of the Binomial density is approximately $\frac{1}{\sqrt{2\pi Ta(1-a)}}$ given by Stirling’s approximation, so we obtain

\[
\int_{2 \frac{1-b}{a} < \frac{\tilde{p}_{i,\ell}}{\tilde{p}_{i,\ell}^{-1}} \leq 2 \frac{b}{a}} \tilde{p}_{i,\ell} \leq \frac{1}{\sqrt{T}},
\]

where we used that $2\pi a(1-a) > 1$ for $a \in [1/4, 3/4]$.

**A.3 Conditional expectations of Gaussian**

**Lemma 11.** Denote by $p$ the density of the Gaussian distribution $\mathcal{N}(\mu, I_n)$. For a unit vector $v \in \mathbb{R}^n$ and $a > 0$, we have

\[
\int_{(y-\mu,v) \geq -a} p(y) (y-\mu) \, dy = \frac{e^{-a^2/2}}{\sqrt{2\pi}} v.
\]
Proof. Note that by translation and rotation invariance of the Gaussian distribution, it suffices to show that
\[ \int_{y_1 \geq -\alpha} p(y_1) y_1 dy_1 = \frac{e^{-\alpha^2/2}}{\sqrt{2\pi}}, \]
since the integral vanishes on all but the first coordinate because of symmetry. A standard computation yields the value on the first coordinate.

A.4 Truncation preserves sub-Gaussianity

We show that truncating a sub-Gaussian random variable preserves its sub-Gaussianity to within a constant factor.

**Lemma 12.** Let \( X \) be a (not necessarily centered) \( \sigma \)-sub-Gaussian random variable, and for some choice \( \lambda \geq 0 \), let \( T_\lambda(X) \) denote its truncation according to equation (35). Then \( T_\lambda(X) \) is \( \sqrt{2}\sigma \)-sub-Gaussian.

**Proof.** The proof follows a symmetrization argument. Let \( X' \) denote an i.i.d. copy of \( X \), and use the shorthand \( Y = T_\lambda(X) \) and \( Y' = T_\lambda(X') \). Let \( \varepsilon \) denote a Rademacher random variable that is independent of everything else. Then \( Y \) and \( Y' \) are i.i.d., and \( \varepsilon(Y - Y') \overset{d}{=} Y - Y' \). Hence we have

\[
\mathbb{E} \left[ e^{t(Y - \mathbb{E}[Y])} \right] = \mathbb{E} \left[ e^{t(Y - \mathbb{E}[Y'])} \right] \\
\leq \mathbb{E}_{Y,Y'} \left[ e^{t(Y - Y')} \right] \\
= \mathbb{E}_{Y,Y',\varepsilon} \left[ e^{t(Y - Y')} \right].
\]

Using the Taylor expansion of \( e^x \), we have

\[
\mathbb{E} \left[ e^{t(Y - \mathbb{E}[Y])} \right] \leq \mathbb{E}_{Y,Y',\varepsilon} \left[ \sum_{i \geq 0} \frac{1}{i!} (t \varepsilon(Y - Y'))^i \right] \\
= \mathbb{E}_{Y,Y'} \left[ \sum_{j \geq 0} \frac{1}{(2j)!} (t(Y - Y'))^{2j} \right],
\]

since only the even moments remain. Finally, since the map \( T_\lambda : \mathbb{R} \to \mathbb{R} \) is 1-Lipschitz, we have \( |Y - Y'| \leq |X - X'| \), and combining this with the fact that \( X - X' \) has odd moments equal to zero yields

\[
\mathbb{E} \left[ e^{t(Y - \mathbb{E}[Y])} \right] \leq \mathbb{E}_{X,X'} \left[ \sum_{j \geq 0} \frac{1}{(2j)!} (t(X - X'))^{2j} \right] \\
= \mathbb{E}_{X,X'} \left[ \sum_{i \geq 0} \frac{1}{i!} (t(X - X'))^i \right] \\
= \mathbb{E}_{X,X'} \left[ e^{t(X - X')} \right] \\
\leq e^{t^2\sigma^2},
\]

41
where the last step follows since the random variable $X - X'$ is zero-mean and $\sqrt{2}\sigma$-sub-Gaussian.

\[ \Box \]

**B Poissonization reduction**

In this section, we show that Poissonization only affects the rates of estimation up to a constant factor. Note that we may assume that $N \geq 4 \log(n_1 n_2)$, since otherwise, all the bounds in the theorems hold trivially.

Let us first show that an estimator designed for a Poisson number of samples may be employed for estimation with a fixed number of samples. Assume that $N$ is fixed, and we have an estimator $\hat{M}_{\text{Poi}}(N)$, which is designed under $N' = \text{Poi}(N)$ observations $\{y_\ell\}_{\ell=1}^N$. Now, given exactly $N$ observations $\{y_\ell\}_{\ell=1}^N$ from the model (1), choose an integer $\tilde{N} = \text{Poi}(N/2)$, and output the estimator

\[
\hat{M}(N) = \begin{cases} 
\hat{M}_{\text{Poi}}(N/2) & \text{if } \tilde{N} \leq N, \\
0 & \text{otherwise}.
\end{cases}
\]

Recalling the assumption $N \geq 4 \log(n_1 n_2)$, we have

\[
\Pr\{\tilde{N} \geq N\} \leq e^{-\frac{N}{2}} \leq (n_1 n_2)^{-2}.
\]

Thus, the error of the estimator $\hat{M}(N)$, which always uses at most $N$ samples, is bounded by

\[
\frac{1}{n_1 n_2} \|\hat{M}(N) - M^*\|_F^2
\]

with probability greater than $1 - (n_1 n_2)^{-2}$, and moreover, we have

\[
\mathbb{E}\left[\frac{1}{n_1 n_2} \|\hat{M}(N) - M^*\|_F^2\right] \leq \mathbb{E}\left[\frac{1}{n_1 n_2} \|\hat{M}_{\text{Poi}}(N/2) - M^*\|_F^2\right] + (n_1 n_2)^{-2}.
\]

We now show the reverse, that an estimator $\hat{M}(N)$ designed using exactly $N$ samples may be used to estimate $M^*$ under a Poissonized observation model. Given $N = \text{Poi}(2N)$ samples, define the estimator

\[
\hat{M}_{\text{Poi}}(2N) = \begin{cases} 
\hat{M}(N) & \text{if } \tilde{N} \geq N, \\
0 & \text{otherwise},
\end{cases}
\]

where in the former case, $\hat{M}(N)$ is computed by discarding $\tilde{N} - N$ samples at random.

Again, using the fact that $N \geq 4 \log(n_1 n_2)$ yields

\[
\Pr\{\tilde{N} \geq N\} \leq e^{-\frac{N}{2}} \leq (n_1 n_2)^{-4},
\]

and so once again, the error of the estimator $\hat{M}_{\text{Poi}}(2N)$ is bounded by

\[
\frac{1}{n_1 n_2} \|\hat{M}(N) - M^*\|_F^2
\]

with probability greater than $1 - (n_1 n_2)^{-4}$. A similar guarantee also holds in expectation.

**References**


