STAT 241/541 Homework 10 Solution

Problem 1:

Using the convolution method,

$$\begin{aligned}
f_{X+Y}(t) &= \int_0^\infty f_X(t-y) f_Y(y) dy \\
&= \int_0^t \frac{\lambda^{\alpha_1}}{\Gamma(\alpha_1)} (t-y)^{\alpha_1 - 1} e^{-\lambda(t-y)} \cdot \frac{\lambda^{\alpha_2}}{\Gamma(\alpha_2)} y^{\alpha_2 - 1} e^{-\lambda y} dy \\
&= \frac{\lambda^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2)} t^{\alpha_1+\alpha_2-1} e^{-\lambda t} \cdot \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \cdot \underbrace{\int_0^t \frac{(t-y)^{\alpha_1-1} y^{\alpha_2-1}}{t^{\alpha_1+\alpha_2-1}} dy}_{\text{(I)}}
\end{aligned}$$

where (I) is equivalent to

$$\int_0^t \left(\frac{t-y}{t}\right)^{\alpha_1-1} \left(\frac{y}{t}\right)^{\alpha_2-1} d\, \frac{y}{t} = \int_0^1 (1-p)^{\alpha_1-1} p^{\alpha_2-1} dp$$

which is the Beta function $Beta(\alpha_1, \alpha_2) = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1+\alpha_2)}$, therefore

$$f_{X+Y}(t) = \frac{\lambda^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2)} t^{\alpha_1+\alpha_2-1} e^{-\lambda t}$$

i.e., $X + Y \sim Gamma(\alpha_1 + \alpha_2, \lambda)$.

Problem 2: Ch 5.2:12

a) $\Pr\left(R = U^2 < \frac{1}{4}\right) = \Pr\left(0 < U < \frac{1}{2}\right) = \frac{1}{2}$

b) $\Pr\left(S < \frac{1}{4}\right) = \Pr\left(U(1-U) < \frac{1}{4}\right) = \Pr\left((U - \frac{1}{2})^2 > 0\right) = 1$

c) $\Pr\left(\frac{U}{1-U} < \frac{1}{4}\right) = \Pr\left(4U < 1 - U\right) = \Pr\left(5U < 1\right) = 0.2$

Problem 3: Ch 5.2:38

We have

$$\begin{aligned}
\Pr(Y_1 \le y_1, Y_2 \le y_2) &= \Pr(\Phi_1(X_1) \le y_1, \Phi_2(X_2) \le y_2) \\
&= \Pr(X_1 \le \Phi_1^{-1}(y_1), X_2 \le \Phi_2^{-1}(y_2)) \\
&= \Pr(X_1 \le \Phi_1^{-1}(y_1)) \Pr(X_2 \le \Phi_2^{-1}(y_2)) \\
&= \Pr(Y_1 \le y_1) \Pr(Y_2 \le y_2)
\end{aligned}$$

Hence, Y_1 and Y_2 are independent.

Problem 4:

a) From the definition of Y, we have $Y = \sqrt{\frac{X}{n-1}}$, where $X \sim \Gamma(\frac{n-1}{2}, \frac{1}{2})$ or χ^2_{n-1}, hence, the density of Y is

$$\begin{aligned} f_Y(y) &= f_X\Big((n-1)y^2\Big) \cdot 2(n-1)y \\ &= \frac{1}{2^{\frac{n-1}{2}}\Gamma(\frac{n-1}{2})}\Big[(n-1)y^2\Big]^{\frac{n-1}{2}-1} \exp\Big[-\frac{1}{2}(n-1)y^2\Big] \cdot 2(n-1)y \\ &= \frac{1}{2^{\frac{n-1}{2}-1}\Gamma(\frac{n-1}{2})}(n-1)^{\frac{n-1}{2}} y^{n-2} \exp\Big[-\frac{1}{2}(n-1)y^2\Big], \quad y \geq 0 \end{aligned}$$

b) We have $T = \frac{Z_1}{Y}$, and Z_1, Y are independent, then the density function for T is

$$\begin{aligned} f_T(t) &= \int_0^\infty y f_{Z_1}(ty) f_Y(y) dy \\ &= \int_0^\infty y \cdot \frac{1}{\sqrt{2\pi}} \exp\Big(-\frac{1}{2}t^2y^2\Big) \cdot \frac{1}{2^{\frac{n-1}{2}-1}\Gamma(\frac{n-1}{2})}(n-1)^{\frac{n-1}{2}} y^{n-2} \cdot \\ &\quad \exp\Big[-\frac{1}{2}(n-1)y^2\Big] dy \\ &= C \cdot \int_0^\infty y^{n-1} \exp\Big[-\underbrace{\frac{1}{2}\Big(t^2+n-1\Big)}_{\text{let this be } A} y^2\Big] dy \\ &= C \cdot \int_0^\infty y^{n-1} \exp(-Ay^2) dy \\ &= C \cdot \frac{1}{2} \cdot A^{-\frac{n}{2}} \cdot \int_0^\infty (Ay^2)^{\frac{n}{2}-1} \exp(-Ay^2)\, d(Ay^2) \\ &= C \cdot \frac{1}{2} \cdot A^{-\frac{n}{2}} \cdot \Gamma(\frac{n}{2}) \end{aligned}$$

where the constant C is

$$\frac{1}{\sqrt{\pi}} \cdot \frac{1}{2^{\frac{n}{2}-1}\Gamma(\frac{n-1}{2})} \cdot (n-1)^{\frac{n-1}{2}}$$

substitute C and A in, we will have

$$f_T(t) = \frac{\Gamma(n/2)}{\sqrt{(n-1)\pi}\Gamma(\frac{n-1}{2})} \cdot \Big(1 + \frac{t^2}{n-1}\Big)^{-\frac{n}{2}}, \quad t \geq 0$$

Problem 5: Ch 8.1:6

Recall Chebyshev's inequality:
For a random variable X, if $E(X) = \mu$ and $Var(X) = \sigma^2$, then

$$\Pr\left(\Big|X - \mu\Big| \geq \varepsilon\right) \leq \frac{\sigma^2}{\varepsilon^2}$$

For the problem, we have $E(\frac{S_n}{n}) = \frac{1}{n} \cdot np = p$ and $Var(\frac{S_n}{n}) = \frac{p(1-p)}{n}$, therefore

$$\Pr\left(\Big|\frac{S_n}{n} - p\Big| \geq \varepsilon\right) \leq \frac{p(1-p)}{n\varepsilon^2}$$