STAT 241/541 Homework 11 Solution

Problem 1: Ch 8.1:14

W.L.O.G., let's assume X is discrete and has distribution function $m(x)$, then

$$\begin{aligned}
E(|X - E(X)|) &= \sum_x |x - E(X)| m(x) \\
&\geq \sum_{|x - E(X)| \geq \varepsilon} |x - E(X)| m(x) \\
&\geq \sum_{|x - E(X)| \geq \varepsilon} \varepsilon m(x) \\
&= \varepsilon \sum_{|x - E(X)| \geq \varepsilon} m(x) \\
&= \varepsilon \Pr(|X - E(X)| \geq \varepsilon).
\end{aligned}$$

Problem 2: Ch 8.2:6

(a) 1 , (b) 1/4 , (c) 1/9 , (d) 1/16 .
The exact values are (a) 0.3173, (b)0.0455, (c) 0.0027, (d) 0.

Problem 3: Ch 9.1:6

We need to choose k so that $\Pr(S_1000 \leq k) \geq 0.99$. This is the same as

$$\Pr\left(S^*_{1000} \leq \frac{k - 500}{\sqrt{250}}\right) \geq 0.99.$$

Thus we want,

$$\frac{k - 500}{15.81} = 2.33$$

This will be true if $k = 537$.

Problem 4: Ch 9.3:14

a) We have $E(X_i) = d$ and $Var(X_i) = 16$ for $i = 1, 2, \ldots, n$. Then

$$E(A_n) = d \text{ and } Var(A_n) = \frac{16}{n}$$

By CLT,

$$\Pr\left(\frac{|A_n - d|}{\frac{1}{\sqrt{n}} \cdot 4} \leq 2\right) \approx 0.95$$

i.e.,

$$\Pr\left(A_n - \frac{8}{\sqrt{n}} \leq d \leq A_n + \frac{8}{\sqrt{n}}\right) \approx 0.95$$

b) Now the observed value for A_n is 23.3 units. The 95% confidence interval for d is $(20.53, 25.87)$.

c) We cannot simply say $\Pr(20.53 \le d \le 25.87) \approx 0.95$, because d is NOT a random variable here. It's just an unknown parameter.

d) From the fact that

$$\Pr\left(A_n - \frac{10.304}{\sqrt{n}} \le d \le A_n + \frac{10.304}{\sqrt{n}}\right) \approx 0.99$$

we can get the 99% confidence interval for d.

Problem 5: Ch 10.1:8

We can derive that $X_i \overset{\text{i.i.d.}}{\sim} Binomial(1, \frac{1}{3})$, for $i = 1, 2, \ldots, n$.

a) $g_1(t) = E(e^{tX_1}) = e^t \cdot \frac{1}{3} + e^0 \cdot \frac{2}{3} = \frac{1}{3}(e^t + 2)$

b) $g_2(t) = E(e^{tS_2}) = E(e^{tX_1})E(e^{tX_2}) = \left(\frac{1}{3}(e^t + 2)\right)^2$

c) $g_n(t) = E(e^{tS_n}) = E(e^{tX_1})E(e^{tX_2}) \cdots E(e^{tX_n}) = \left(\frac{1}{3}(e^t + 2)\right)^n$

Problem 6: Ch 10.3:8

Let $X_i \overset{\text{i.i.d.}}{\sim} U(a, b)$ for $i = 1, 2, \ldots, n$, then

a) $g(t) = E(e^{tX_1}) = \displaystyle\int_a^b e^{tx} \cdot \frac{1}{b-a} dx = \frac{e^{bt} - e^{at}}{t(b-a)}$

b) $g(t) = E(e^{tX_1})E(e^{tX_2}) = \left(\dfrac{e^{bt} - e^{at}}{t(b-a)}\right)^2$

c) $g(t) = E(e^{tX_1})E(e^{tX_2}) \cdots E(e^{tX_n}) = \left(\dfrac{e^{bt} - e^{at}}{t(b-a)}\right)^n$

d) $g(t) = E(e^{\frac{1}{n}tX_1})E(e^{\frac{1}{n}tX_2}) \cdots E(e^{\frac{1}{n}tX_n}) = \left(\dfrac{n(e^{bt/n} - e^{at/n})}{t(b-a)}\right)^n$

e) $g(t) = E(e^{\frac{1}{\sqrt{n\sigma^2}}tS_n} \cdot e^{-\frac{n\mu}{\sqrt{n\sigma^2}}t}) = e^{-\frac{n\mu}{\sqrt{n\sigma^2}}t}\left(\dfrac{\sqrt{n\sigma^2}(e^{bt/\sqrt{n\sigma^2}} - e^{at/\sqrt{n\sigma^2}})}{t(b-a)}\right)^n$