

# STAT 241/541 Homework 12 Solution

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## Problem 1: Ch 10.3:10

a) Since the density function is an even function over the range  $(-\infty, +\infty)$ , the mean is 0. This can also be seen from the following, i.e., by symmetry, the integrand is an odd function and the range of the integral is also symmetric around 0, therefore,

$$E(X) = \int_{-\infty}^{+\infty} x \cdot \frac{1}{2} e^{-|x|} dx = 0.$$

We can calculate the variance by using the fact  $Var(X) = E(X^2) - [E(X)]^2$ , and

$$\begin{aligned} E(X^2) &= \int_{-\infty}^{+\infty} x^2 \frac{1}{2} e^{-|x|} dx \\ &= 2 \int_0^{\infty} x^2 \frac{1}{2} e^{-x} dx = \int_0^{\infty} x^{3-1} e^{-x} dx \\ &= \Gamma(3) = 2! = 2 \end{aligned}$$

Therefore,  $Var(X) = 2 - 0 = 2$ .

b) The moment generating function for  $X_1$  is

$$\begin{aligned} g_{X_1}(t) &= E(e^{X_1 t}) = \int_{-\infty}^{+\infty} e^{x_1 t} \frac{1}{2} e^{-|x_1|} dx_1 \\ &= \frac{1}{2} \left[ \int_{-\infty}^0 e^{(t+1)x_1} dx_1 + \int_0^{+\infty} e^{(t-1)x_1} dx_1 \right] \\ &= \frac{1}{2} \left[ \frac{1}{t+1} + \frac{1}{t-1} \right], \text{ where } |t| < 1 \\ &= \frac{1}{1-t^2}, \quad |t| < 1 \end{aligned}$$

Similarly, we can get the other moment generating functions, for  $|t| < 1$

$$g_{S_n}(t) = \left( \frac{1}{1-t^2} \right)^n, \quad g_{A_n}(t) = \left( \frac{1}{1-\frac{t^2}{n}} \right)^n, \quad g_{S_n^*}(t) = \left( \frac{1}{1-\frac{t^2}{2n}} \right)^n.$$

- c)  $g_{S_n^*}(t) \rightarrow e^{\frac{t^2}{2}}$  as  $n \rightarrow \infty$ .  
d)  $g_{A_n}(t) \rightarrow 1$  as  $n \rightarrow \infty$ .

**Problem 2:**

The joint density function for  $(U_1, U_2)$  is:

$$\varphi(u_1, u_2) = \frac{1}{8\pi} \exp\left(-\frac{(u_1 - 2)^2 + (u_2 + 2)^2}{8}\right)$$

**Problem 3:**

The joint density function for  $(U_1, U_2)$  is:

$$\varphi(u_1, u_2) = \frac{1}{2\pi} \exp\left(-\frac{(u_1 - \sqrt{2})^2 + u_2^2}{2}\right)$$