STAT 241/541 Homework 1 Solution

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Problem 1: Ch1.2:14

(a) The possible values that Y may take are 2, 3, 4 and 5. Therefore, the distribution function of Y is

$$m_Y(2) = 1/5, \quad m_Y(3) = 1/5, \quad m_Y(4) = 2/5, \quad m_Y(5) = 1/5.$$

(b) The possible values for Z are 0, 1, and 4. We have

$$m_Z(0) = 1/5, \quad m_Z(1) = m_X(-1) + m_X(1) = 3/5, \quad m_Z(4) = 1/5.$$

Problem 2: 18

(a) The right-hand side of the sum of the probabilities of all outcomes occurring in the left-hand side plus some more because of duplication. Here is one approach to prove it:

By theorem 1.4 in the book, $\Pr(A \cup B) \leq \Pr(A) + \Pr(B)$. Let event $E_m = A_{m+1} \cup A_{m+2} \cup \cdots \cup A_n$, for $m = 1, 2, \ldots, n$, then

$$\begin{aligned}
\text{L.H.S.} &= \Pr(A_1 \cup E_1) \\
&\leq \Pr(A_1) + \Pr(E_1) \\
&= \Pr(A_1) + \Pr(A_2 \cup E_2) \\
&\leq \Pr(A_1) + \Pr(A_2) + \Pr(E_2) \\
&\quad \cdots \\
&\leq \Pr(A_1) + \Pr(A_2) + \ldots + \Pr(A_{n-1}) + \Pr(E_{n-1}) \\
&= \text{R.H.S.}
\end{aligned}$$

(b)

$$1 \geq \Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$

Problem 3: 20

For uniform distribution, we will have to have the same probability assigned to all outcomes. If this probability is 0, the sum of the probabilities would be 0 so that $\Pr(\Omega) = 0$ instead of 1 as it should be. If this common probability is $a > 0$, then the sum of all the probabilities of the first n outcomes would be na and for large enough n this would be greater than 1, contradicting the requirement that the sum of the probabilities for all possible outcomes should be 1.

Problem 4: 22

The sample space $\Omega = \{1, 2, 3, \ldots\}$ and the distribution function is

$$m(n) = \frac{1}{6} \cdot \left(\frac{5}{6}\right)^{n-1}, \quad n = 1, 2, 3, \ldots$$

Now if $0 < x < 1$,

$$\sum_{j=0}^{\infty} x^j = \frac{1}{1-x} \ .$$

Hence

$$(1/6) \sum_{j=0}^{\infty} (5/6)^{n-1} = (1/6) \cdot \frac{1}{1 - 5/6} = 1.$$

Problem 5: 26

The probability that the two cards are of the same rank is $\frac{52 \cdot 3}{52 \cdot 51} = \frac{1}{17}$. Thus $2x + \frac{1}{17} = 1$ giving $x = \frac{8}{17}$.

Problem 6: 28

(a) 1/3

(b) $P_3(N) = \frac{[\frac{N}{3}]}{N}$, where $[\frac{N}{3}]$ is the greatest integer less than or equal to $\frac{N}{3}$. Note that

$$\frac{N}{3} - 1 \leq \left[\frac{N}{3}\right] \leq \frac{N}{3} \ .$$

From this we see that

$$P_3 = \lim_{N \to \infty} P_3(N) = \frac{1}{3} \ .$$

(c) If A is a finite set with K elements then

$$\frac{A(N)}{N} \leq \frac{K}{N}$$

, So

$$\lim_{N \to \infty} \frac{A(N)}{N} = 0 \ .$$

On the other hand, if A is the set of all positive integers, then

$$\lim_{N \to \infty} \frac{A(N)}{N} = \lim_{N \to \infty} \frac{N}{N} = 1 \ .$$

(d) Let $N_k = 10^k - 1$. Then the integers between $N_{k-1}+1$ and N_k have exactly k digits. Thus, if k is odd, then

$$A(N_k) = (N_k - N_{k-1}) + (N_{k-2} - N_{k-3}) + \dots ,$$

while if k is even, then

$$A(N_k) = (N_{k-1} - N_{k-2}) + (N_{k-3} - N_{k-4}) + \dots .$$

Thus, if k is odd, then

$$A(N_k) \geq (N_k - N_{k-1}) = 9 \cdot 10^{k-1} ,$$

while if k is even, then

$$A(N_k) \leq N_{k-1} < 10^{k-1} .$$

So, if k is odd, then

$$\frac{A(N_k)}{N_k} \geq \frac{9 \cdot 10^{k-1}}{10^k - 1} > \frac{9}{10} ,$$

while if k is even, then

$$\frac{A(N_k)}{N_k} < \frac{10^{k-1}}{10^k - 1} < \frac{2}{10} .$$

Therefore, $\lim_{N\to\infty} \frac{A(N)}{N}$ does not exist.