STAT 241/541 Homework 2 Solution

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Problem 1: Ch2.2:2

(a) $c = 1/48$.

(b) $\Pr(E) = \int_a^b \frac{1}{48} x dx = \frac{1}{96}(b^2 - a^2)$.

(c) $\Pr(X > 5) = \int_5^{10} \frac{1}{48} x dx = \frac{75}{96}$, $\Pr(X < 7) = \int_2^7 \frac{1}{48} x dx = \frac{45}{96}$,

and $\Pr(X^2 - 12X + 35 > 0) = 1 - \Pr(5 < X < 7) = 1 - \frac{24}{96} = \frac{3}{4}$.

Problem 2: Ch2.2:4

(a) .04, (b) .36, (c).25, (d) .09.

Problem 3: Ch2.2:6

(a) Let Y be the time until a new light bulb burns out, and Y has the density function

$$f_Y(y) = \lambda e^{-\lambda y} \ , \quad \text{for } y \in [0, \infty)$$

Then,

$$\begin{aligned} & \Pr(\text{the bulb will not burn out before } T \text{ hours}) \\ = \ & \Pr(Y > T) = \int_T^\infty f_Y(y) dy = \int_T^\infty -.01 e^{-.01y} dy \\ = \ & e^{-.01T}. \end{aligned}$$

(b) Set $e^{-.01T} = 1/2$ and we get $T = 100 \log(2) \approx 69.3$.

Problem 4: Ch2.2:8

(a) 1/8 , (b) $1/2 + \int_{1/2}^1 \frac{1}{2x} dx = \frac{1}{2}(1 + \log(2))$, (c) 3/4, (d) 1/4,

(e) 3/4, (f) 1/4, (g) 1/8, (h) $\pi/8$, (i)$\pi/4$.

Problem 4: Ch2.2:12

According to the hint, each of the 3 pieces should be larger than 1/2. We provide two methods to solve this problem.

Method 1: First we break the stick into 2 pieces and let X be the length of the longer piece. Then, we break the longer piece into 2 and let Y be the length of one of them. So we have three pieces of length $1-X$, Y, and $X-Y$. Be careful here we have the implicit requirement $X > Y$, therefore our sample space is different from that of problem 2.2:8. The sample space here will be all the points in the set $\{(x, y) : x \in [0, 1], y \in [0, 1], x > y\}$, solve

$$\begin{cases} X - Y < 1/2, \\ 1 - X < 1/2, \\ Y < 1/2 \end{cases}$$

and we can get the probability is $\frac{1}{8}/\frac{1}{2} = 1/4$.

Method 2: Let X, Y, and $1 - X - Y$ be the lengths of the three pieces respectively. Note X and Y should satisfy $X + Y < 1$. Then the sample space is $\{(x, y) : x \in [0, 1], y \in [0, 1], x + y < 1\}$, hence we have

$$\begin{cases} X < 1/2, \\ Y < 1/2, \\ 1 - X - Y < 1/2, \end{cases}$$

again, this deduces the probability should be 1/4.

Extra Problem:

(a) $F_Y(y) = \Pr(Y \le y) = \Pr(\frac{1}{X} \le y) = \Pr(X \ge \frac{1}{y}) = 1 - \frac{1}{y}$.
Therefore, the probability density function of Y is

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{1}{y^2}, \quad y > 1$$

and $E(Y) = \int_1^\infty \frac{1}{y^2} dy = \infty$.
(b) The c.d.f of Y

$$F_Y(y) = \Pr(Y \le y) = \Pr(\tan \frac{\pi X}{2} \le y) = \Pr(X \le \frac{2}{\pi} \arctan y) = \frac{2}{\pi} \arctan y.$$

Then, the p.d.f is

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{2}{\pi} \cdot \frac{1}{1+y^2}, \quad y > 0$$

Therefore, $E(Y) = \int_0^\infty \frac{2}{\pi} \cdot \frac{1}{1+y^2} dy = \infty$.