Stat 241 Homework 3 Solution

Section 4.1

4). The probabilities are

$$\Pr(\text{heart} \mid \text{red}) = \frac{\Pr(\text{heart} \cap \text{red})}{\Pr(\text{red})} = \frac{13/52}{26/52} = \frac{1}{2}$$

$$\Pr(> 10 \mid \text{heart}) = \frac{\Pr(> 10 \cap \text{heart})}{\Pr(\text{heart})} = \frac{4/52}{13/52} = \frac{4}{13}$$

$$\Pr(\text{Jack} \mid \text{red}) = \frac{\Pr(\text{Jack} \cap \text{red})}{\Pr(\text{red})} = \frac{2/52}{26/52} = \frac{1}{13}$$

8). $\Pr(A \cap B \cap C) = \Pr(\{a\}) = 1/8$,
$\Pr(A) = \Pr(B) = \Pr(C) = 1/2$.
Thus while

$$\Pr(A \cap B \cap C) = \Pr(A)\Pr(B)\Pr(C) = \frac{1}{8}$$

we have

$$\Pr(A \cap B) = \Pr(B \cap C) = \Pr(A \cap C) = \frac{5}{16}$$

and

$$\Pr(A)\Pr(B) = \Pr(B)\Pr(C) = \Pr(A)\Pr(C) = \frac{1}{4}$$

Therefore no two of these events are independent.

18). We have

$$\begin{aligned}
\Pr(+) &= \Pr(+ \mid d_1)\Pr(d_1) + \Pr(+ \mid d_2)\Pr(d_2) + \\
&\quad \Pr(+ \mid d_3)\Pr(d_3) \\
&= 0.8 \times \frac{1}{3} + 0.6 \times \frac{1}{3} + 0.4 \times \frac{1}{3} \\
&= 0.6
\end{aligned}$$

Then

$$\Pr(d_1 \mid +) = \frac{\Pr(+ \mid d_1)\Pr(d_1)}{\Pr(+)} = \frac{0.8 \times 1/3}{0.6} = \frac{4}{9}$$

Similarly, we have $\Pr(d_2 \mid +) = 1/3$ and $\Pr(d_3 \mid +) = 2/9$.

28). The most obvious objection is the assumption that all of the events in question are independent. A more subtle objection is that, since Los Angeles is so large, it is reasonable to ask for the probability that there is at least two couples with the same description, given that there is one such couple. This probability is not so small. Let's assume there are $n = 5,000,000$ couples in L.A. The probability of a randomly chosen couple has such characteristics is $p = 1/12,000,000$.

Let N be the number of such couple in L.A., then

$$\Pr(N = i) = \binom{n}{i} p^i (1-p)^{n-i}, \quad \text{for } i = 0, 1, \ldots, n$$

So the conditional probability

$$\Pr(N \geq 2 \mid N \geq 1) = \frac{\Pr(N \geq 2)}{\Pr(N \geq 1)} = \frac{1 - \Pr(N = 0) - \Pr(N = 1)}{1 - \Pr(N = 0)}$$

After we substitute $\Pr(N = 0) \approx 0.659$ and $\Pr(N = 1) \approx 0.275$, we got

$$\Pr(N \geq 2 \mid N \geq 1) \approx 0.194$$

Hence, the probability of finding more such couple given there is at least one is not at all small.

Section 4.3

2). E: the event that the hand has at least one ace;
F: the event that the hand has at least two aces;
S: the event that the hand has the ace of hearts.

8-card case (4 aces and 4 kings):

$$\Pr(F \mid E) = \frac{\Pr(F \cap E)}{\Pr(E)} = \frac{\Pr(F)}{\Pr(E)} = \frac{\binom{4}{2}}{\binom{8}{2} - \binom{4}{2}} = \frac{3}{11}$$

$$\Pr(F \mid S) = \frac{\Pr(F \cap S)}{\Pr(S)} = \frac{\binom{3}{1}}{\binom{8}{2} - \binom{7}{2}} = \frac{3}{7}$$

52-card case:

$$\Pr(F \mid E) = \frac{\Pr(F \cap E)}{\Pr(E)} = \frac{\Pr(F)}{\Pr(E)} = \frac{\binom{4}{2}\binom{48}{11} + \binom{4}{3}\binom{48}{10} + \binom{4}{4}\binom{48}{9}}{\binom{52}{13} - \binom{48}{13}} \approx 0.3696$$

$$\Pr(F \mid S) = \frac{\Pr(F \cap S)}{\Pr(S)} = \frac{\binom{3}{1}\binom{48}{11} + \binom{3}{2}\binom{48}{10} + \binom{3}{3}\binom{48}{9}}{\binom{52}{13} - \binom{51}{13}} \approx 0.5612$$

Extra problem:

Let $P_1 = \Pr(\text{win the cacino} \mid \text{has 1 unit of money at the first bet})$
Harry derives the formula for P_1 in his lecture as follows:

$$P_1 = \frac{1 - \frac{1-p}{p}}{1 - (\frac{1-p}{p})^{n+1}}$$

where p in our case is 0.51 and n is the amount of money Dan need to win. Since $0 < \frac{1-p}{p} = \frac{0.49}{0.51} < 1$, we have

$$P_1 = \frac{1 - \frac{1-p}{p}}{1 - (\frac{1-p}{p})^{n+1}} > 1 - \frac{1-p}{p} = 2 - \frac{1}{p} > 0$$

So if Dan bets m times, the probability that he will lose all the time is:

$$0 < (1 - P_1)^m < 1$$

So, if Dan wants to win the casino with probability 1, he would like to make the above probability as small as possible, i.e., he wants to play as many as possible.

Here is a strategy for Dan is he has only 1 dollar. We know

$$1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$$

and the casino allows Dan to bet any amount of money. So what Dan has to do is to bet $\frac{1}{2}$ first time, $\frac{1}{4}$ at the second time, etc. In this way, Dan can play infinite number of times, i.e. $m \to \infty$, and the probability that Dan will lose all his money is $\to 0$ and he will surely win the casino.