

Stat 241/541 Homework 5 Solution

Section 3.1: 4

4. At this writing, 37 Presidents have died. The probability that no two people from a group of 37 (all of whom are dead) died on the same day is

$$\frac{365 \times 364 \times \cdots \times (365 - 37 + 1)}{365^{37}} \approx 0.15 .$$

Thus, the probability that at least two died on the same day is 0.85.

Section 3.2: 7, 18, 32

7. We have

$$\begin{aligned} \frac{b(n, p, j)}{b(n, p, j-1)} &= \frac{\binom{n}{j} p^j q^{n-j}}{\binom{n}{j-1} p^{j-1} q^{n-j+1}} \\ &= \frac{n!}{j!(n-j)!} \frac{(n-j+1)!(j-1)!}{n!} \frac{p}{q} \\ &= \frac{(n-j+1)p}{j} \frac{p}{q}, \quad \text{where } q = 1-p \end{aligned}$$

but $\frac{(n-j+1)p}{j} \frac{p}{q} \geq 1$ if and only if $j \leq p(n+1)$, and so $j = \lfloor p(n+1) \rfloor$ gives $b(n, p, j)$ its largest value. If $p(n+1)$ is an integer there will be two possible values of j , namely $j = p(n+1)$ and $j = p(n+1) - 1$. Because when $j = p(n+1)$, $b(n, p, j) = b(n, p, j-1)$.

18. $\Pr(\text{no student gets 2 or fewer correct}) = b(340, 7/128, 0) \approx 4.96 \cdot 10^{-9}$;
 $\Pr(\text{no student gets 0 correct}) = b(340, 1/1024, 0) \approx 0.717$.
So Prosser is right to expect at least one student with 2 or fewer correct, but Crowell is wrong to expect at least one student with none correct.

32. Suppose we assign x people to the east side and $n-x$ to the west side of the mountain. Then we want to maximize the probability:

$$\begin{aligned} \Pr(E) &= \Pr(\text{find the boy}) \\ &= \Pr(\text{find the boy} \mid \text{east})p + \Pr(\text{find the boy} \mid \text{west})(1-p) \\ &= [1 - (1-u)^x]p + [1 - (1-u)^{n-x}](1-p) \end{aligned} \quad (1)$$

Take derivative of (1) w.r.t x and set it to zero, then we can get

$$x_{\text{critical}} = \frac{n}{2} - \frac{\ln(\frac{p}{1-p})}{2 \ln(1-u)}$$

If $u = 1$, you only need to be sure to send at least one to each side. If $u = 0$, it doesn't matter what you do. If $0 < u < 1$, x should be the **nearest integer** to $x_{critical}$.

Extra problem:

Suppose one player gets 1 dollar if she wins the game and loses 1 dollar otherwise. let X_i and Y_i be the net gain for the players in the i -th game, respectively. Then,

$$X_i, Y_i = \begin{cases} 1, & \text{with prob'y } \frac{1}{2} \\ -1, & \text{with prob'y } \frac{1}{2} \end{cases}$$

Let $S_n = \sum_{i=1}^n X_i$ and $T_n = \sum_{i=1}^n Y_i$ be the total money of the two players after the n -th game. Then the players will quit if and only if

$$S_n = T_n, \quad \text{for some } n$$

From the setting, we can see that (S_n, T_n) is a 2 dimensional random walk. Harry has provided the fact that for any random walk in less than 3 dimension, it is for sure that the random walk will return to the starting point, i.e.,

$$\Pr(S_n = T_n = 0, \text{ for some } n) \equiv 1$$

Since $\{S_n = T_n, \text{ for some } n\} \supset \{S_n = T_n = 0, \text{ for some } n\}$, we have

$$\begin{aligned} \Pr(\text{quit the game}) &= \Pr(\{S_n = T_n, \text{ for some } n\}) \\ &\geq \Pr(\{S_n = T_n = 0, \text{ for some } n\}) \\ &= 1 \end{aligned}$$

Hence, the players will surely quit.

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