Stat 241/541 Homework 6 Solution

Problem 1:

Let P_0 be the probability that the random walk never returns to 0. Let E be the event that the random walk returns to 0 finitely often. Then for our one dimensional random walk,

$$\begin{aligned}\Pr(S_{2n}=0) &= \Pr(\text{ a Binomial}\,(2n,p) \text{ takes the value } n) \\ &= \binom{2n}{n} p^n (1-p)^n \\ &\sim \frac{1}{\sqrt{\pi n}} \Big[4p(1-p)\Big]^n\end{aligned}$$

Since $p \neq 1/2$, we have $0 \leq 4p(1-p) < 1$. Let a_n denote $\Pr(S_{2n}=0)$, and by ratio test:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 4p(1-p) < 1$$

The ratio is less than 1, so the series converges, i.e.,

$$\begin{aligned}\sum_{n=0}^{\infty} \Pr(S_{2n}=0) &= E\Big(\sum_{n=0}^{\infty} I_{\{S_{2n}=0\}}\Big) \\ &= \text{Expected number of returns to 0} \\ &< \infty \qquad (1)\end{aligned}$$

Now suppose $P_0 = 0$, then $P(E) = 0$, i.e.,

$$\Pr(\text{random walk returns to 0 infinitely often}) = 1,$$

which means the expected number of returns to 0 is infinite. This contradicts with (1). Therefore, P_0 must be positive.

Problem 2:

It's easy to see that the state space for the 2-D random walk, i.e., the set of points that the random walk will cover, is

$$E = \{(i,j) : i-j = 2k,\ k \in \mathbb{Z}\}$$

So what we need to show is that the random walk will visit such points for sure. Without loss of generality , let's assume that we starts at the origin $(0,0)$. As for general (i,j), where $i-j$ is even, we can just choose our starting point at, say, $(i-1, j-1)$. One can see that for the 1st step,

$$\Pr\Big(\text{the r.w. will not go to } (1,1)\Big) = \frac{3}{4}$$

We have known that for 2-D random walk, it's for sure that the r.w. will come back to the starting point, and

$$\Pr\Big(\text{the r.w. will not go to } (1,1) \text{ for a second time}\Big) = \Big(\frac{3}{4}\Big)^2$$

So eventually, one can get

$$\Pr\Big(\text{the r.w. never goes to } (1,1)\Big) = \Big(\frac{3}{4}\Big)^\infty = 0$$

which means the r.w. will surely go to (1,1). And for points like (1,-1), (-1,1) and (-1,-1), the arguments are the same. One can iterate the above step from the 4 points by choosing them as starting points and prove that all the points in E will be covered for sure.

Problem 3:

Let's number the letters by the set $\{1, 2, \ldots, n\}$. Let S be the number of correct matches. Then,

$$\begin{aligned}\Pr(S=k) &= \sum_{i_1,i_2,\ldots,i_k} \Pr\Big(\{\text{letters numbered } i_1, i_2, \ldots, i_k \text{ are} \\ &\qquad\qquad \text{correctly matched and the rest are not}\}\Big)\end{aligned}$$

From Harry's lecture notes we know that,

$$\begin{aligned}P &= \Pr\Big((n-k) \text{ incorrect matches}\Big) = \Pr\Big(\text{messed up all } (n-k) \text{ hats}\Big) \\ &= 1 - \Big(1 - \frac{1}{2!} + \frac{1}{3!} - \cdots + (-1)^{(n-k-1)}\frac{1}{(n-k)!}\Big) \\ &= 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^{(n-k)}\frac{1}{(n-k)!}\end{aligned}$$

so we have

$$\begin{aligned}\Pr(S=k) &= \sum_{i_1,\ldots,i_k}\Big[\frac{1}{n(n-1)\cdots(n-k)}\cdot P\Big] \\ &= \binom{n}{k}\frac{(n-k)!}{n!}\cdot P \\ &= \frac{1}{k!}\cdot P \qquad \text{for } k = 0, 1, \ldots, n\end{aligned}$$

Therefore,

$$\Pr(S=k) = \frac{1}{k!}\Big(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^{(n-k)}\frac{1}{(n-k)!}\Big) \quad \text{for } k = 0, 1, \ldots, n$$

For fixed k, as $n \to \infty$, the probability converges to

$$\frac{1}{k!}\Big(1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots\Big) = \frac{e^{-1}}{k!},$$

which is the probability $Y = k$ if Y has a Poisson(1) distribution.

Problem 4:

According to Harry's hint, we want to find $\Pr(C \mid X = 4)$. By law of total probability

$$\Pr(C \mid X = 4) = \Pr(C \cap G \mid X = 4) + \Pr(C \cap S \mid X = 4)$$

and we have,

$$\begin{aligned} \Pr(C \cap G \mid X = 4) &= \Pr(G \mid X = 4) \Pr(C \mid \{X = 4\} \cap G) \\ \Pr(C \cap S \mid X = 4) &= \Pr(S \mid X = 4) \Pr(C \mid \{X = 4\} \cap S) \end{aligned}$$

If we know the type of candidate, the $\{X = 4\}$ information becomes irrelevant. Hence we have,

$$\Pr(C \mid X = 4) = \frac{1}{3} \Pr(G \mid X = 4) + \frac{1}{2} \Pr(S \mid X = 4)$$

One can calculate,

$$\begin{aligned} \Pr(X = 4 \mid G) &= \binom{6}{4} (1/3)^4 (2/3)^2 = \binom{6}{4} \frac{4}{729} \\ \Pr(X = 4 \mid S) &= \binom{6}{4} (1/2)^4 (1/2)^2 = \binom{6}{4} \frac{1}{64} . \end{aligned}$$

Apply the usual splitting/conditioning argument,

$$\begin{aligned} \Pr(G \mid X = 4) &= \frac{\Pr(X = 4 \mid G) \Pr(G)}{\Pr(X = 4)} \\ &= \frac{\Pr(X = 4 \mid G) \Pr(G)}{\Pr(X = 4 \mid G) \Pr(G) + \Pr(X = 4 \mid S) \Pr(S)} \\ &= \frac{\binom{6}{4} 4/729(0.3)}{\binom{6}{4} 4/729(0.3) + \binom{6}{4} 1/64(0.7)} \\ &\approx 0.131 \end{aligned}$$

There is no need to repeat the calculation for the other conditional probability, because

$$\Pr(S \mid X = 4) = 1 - \Pr(G \mid X = 4) \approx 0.869$$

Therefore,

$$\Pr(C \mid X = 4) = \frac{1}{3} \times 0.131 + \frac{1}{2} \times 0.869 \approx 0.478$$

Section 5.1: 2

The total number of permutations is $n!$ and X is just one realization of such permutations and has probability $1/n!$. Hence, X is uniformly distributed.