

STAT 241/541 Homework 7 Solution

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Section 6.1: 20, 30, 36

20 (a) The expected value of winnings:

$$\begin{aligned} E(X) &= \frac{1}{2} \cdot 2 + \left(\frac{1}{2}\right)^2 \cdot 2^2 + \left(\frac{1}{2}\right)^3 \cdot 2^3 + \dots \\ &= 1 + 1 + 1 + \dots \\ &= \infty \end{aligned}$$

This means that if we could play the game, it would be favorable no matter how much we pay to play it. However, we cannot realize this game, since it requires arbitrarily large amounts of money.

(b)

$$\begin{aligned} E(X) &= \frac{1}{2} \cdot 2 + \left(\frac{1}{2}\right)^2 \cdot 2^2 + \dots + \left(\frac{1}{2}\right)^{10} \cdot 2^{10} + 2^{10} \left(\frac{1}{2^{11}} + \frac{1}{2^{12}} + \dots\right) \\ &= 10 + \frac{1}{2} + \frac{1}{2^2} + \dots = 11 \end{aligned}$$

(d) If the utility of n dollars is \sqrt{n} , then the expected utility of the payment is given by

$$\sum_{i=1}^{\infty} \frac{1}{2^i} \sqrt{2^i} = \frac{1}{\sqrt{2}-1}$$

If the utility of n dollars is $\log n$, then the expected utility of the payment is given by

$$\sum_{i=1}^{\infty} \frac{1}{2^i} \log(2^i) = 2 \log 2$$

30 (a) $\Pr(X_k = j) = \Pr\left\{(j-1) \text{ boxes have old pics and the } j\text{th box has a new pic}\right\}$

$$= \left(\frac{k-1}{n}\right)^{j-1} \left(\frac{n-k+1}{n}\right)$$

and so X_k has a geometric distribution with $p = (n-k+1)/n$.

(c) The expected time for getting the first half of the players is

$$\begin{aligned} E(X_1) + \dots + E(X_n) &= \frac{2n}{2n-1+1} + \frac{2n}{2n-2+1} + \dots + \frac{2n}{2n-n+1} \\ &= 2n \left(\frac{1}{2n} + \frac{1}{2n-1} + \frac{1}{n+1} \right) \end{aligned}$$

The expected time for getting the second half of the players is:

$$\begin{aligned} E(X_{n+1}) + \cdots + E(X_{2n}) &= \frac{2n}{2n - (n+1) + 1} + \cdots + \frac{2n}{2n - 2n + 1} \\ &= 2n \left(\frac{1}{n} + \frac{1}{n-1} + \frac{1}{1} \right) \end{aligned}$$

(d) We have known that

$$1 + \frac{1}{2} + \cdots + \frac{1}{n} \sim \log n + .5772 + \frac{1}{2n} \quad (1)$$

$$1 + \frac{1}{2} + \cdots + \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n} \sim \log 2n + .5772 + \frac{1}{4n} \quad (2)$$

Using $2n$ multiply (2) - (1) yields

$$\begin{aligned} 2n \left(\frac{1}{2n} + \cdots + \frac{1}{n+1} \right) &\sim 2n \left(\log 2n + \frac{1}{4n} - \log n - \frac{1}{2n} \right) \\ &= 2n \left(\log 2 - \frac{1}{4n} \right) \\ &= 2n \log 2 - \frac{1}{2} \\ 2n \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} \right) &\sim 2n \left(\log n + .5772 + \frac{1}{2n} \right) \end{aligned}$$

- 36 First let's number the yellow balls using the set $\{1, 2, \dots, c\}$. Let Z be the number of yellow balls drawn from the urn and random variable X_i be such that

$$X_i = \begin{cases} 1, & \text{if the } i\text{th yellow ball is drawn;} \\ 0, & \text{otherwise.} \end{cases}$$

Then we have

$$Z = X_1 + X_2 + \cdots + X_c$$

Note that for any ball in the urn, the probability of that ball being drawn when drawing k out of $c + d$ balls is $k/(c + d)$, i.e.,

$$\Pr(X_i = 1) = \frac{k}{c + d} = p$$

So we have $E(X_i) = 1 \cdot p + 0 \cdot (1 - p) = p$, and

$$\begin{aligned} E(Z) &= E(X_1) + E(X_2) + \cdots + E(X_c) \\ &= cp = \frac{ck}{c + d} \end{aligned}$$

Actually Z in this problem follows a *hypergeometric distribution* with p.m.f

$$\Pr(Z = i) = \frac{\binom{c}{i} \binom{d}{k-i}}{\binom{c+d}{k}}$$

An alternative method is to derive $E(Z)$ by computing $\sum i \Pr(Z = i)$.

Section 6.2: 18, 20

18 (a) $E(\bar{x}) = \frac{1}{n} \sum_{i=1}^n E(x_i) = \frac{1}{n} \cdot n\mu = \mu.$

(b) We have

$$E\left((\bar{x} - \mu)^2\right) = V(\bar{x}),$$

which was shown to equal σ^2/n in Theorem 6.9.

(c) We have from the hint:

$$\sum_{i=1}^n (x_i - \bar{x})^2 = \sum_{i=1}^n (x_i - \mu)^2 - n(\bar{x} - \mu)^2.$$

Thus,

$$\begin{aligned} E(s^2) &= \frac{1}{n} E\left(\sum_{i=1}^n (x_i - \bar{x})^2\right) \\ &= \frac{1}{n} \left(E\left(\sum_{i=1}^n (x_i - \mu)^2\right) - nE(\bar{x} - \mu)^2 \right) \\ &= \frac{1}{n} \left(n\sigma^2 - n \frac{\sigma^2}{n} \right) = \frac{n-1}{n} \sigma^2, \end{aligned}$$

where we have used the definition of the variance and part (b) to obtain the penultimate expression.

(d) Since the expectation operator is linear, and the ‘new’ s^2 is $n/(n-1)$ times the ‘old’ s^2 , the new s^2 has expectation

$$\frac{n}{n-1} \frac{n-1}{n} \sigma^2 = \sigma^2.$$

20 (a) $E(\bar{\mu}) = E\left(\omega X_1 + (1-\omega)X_2\right) = \omega E(X_1) + (1-\omega)E(X_2) = \omega\mu + (1-\omega)\mu = \mu$

(b) Since X_1 and X_2 are independent, we have

$$\begin{aligned} Var(\bar{\mu}) &= Var\left(\omega X_1 + (1-\omega)X_2\right) \\ &= \omega^2 Var(X_1) + (1-\omega)^2 Var(X_2) \\ &= \omega^2 \sigma_1^2 + (1-\omega)^2 \sigma_2^2 \\ &= (\sigma_1^2 + \sigma_2^2)\omega^2 - 2\sigma_2^2\omega + \sigma_2^2 \end{aligned}$$

and we know $\omega^* = \sigma_2^2/(\sigma_1^2 + \sigma_2^2)$ is the minimizer to the above quadratic function.