STAT 241/541 Homework 8 Solution

Section 5.1: 10, 24, 32, 44

10 (a) The probability of $X = k$ is:

$$\Pr(X = k) = \frac{\binom{N}{k}\binom{N-k}{n_1-k}\binom{N-n_1}{n_2-k}}{\binom{N}{n_1}\binom{N}{n_2}} = \frac{(N-n_1)!(N-n_2)!n_1!n_2!}{N!(N-n_1-n_2+k)!(n_1-k)!(n_2-k)!k!}$$

(b) Let p_N denote the probability $X = n_{12}$, given that the total population is N. Then p_N equals the expression in the answer to part (a), with k replaced by n_{12}. According to the hint, we consider the ratio:

$$\frac{p_N}{p_{N-1}} = \frac{(N-n_1)(N-n_2)}{N(N-n_1-n_2+n_{12})} = \frac{N^2-(n_1+n_2)N+n_1n_2}{N^2+(n_{12}-n_1-n_2)N}$$

By setting the above ratio *greater than or equal to* 1, we obtain

$$N \le \frac{n_1n_2}{n_{12}}$$

Therefore, $N = \left[\frac{n_1n_2}{n_{12}}\right]$ will give us the maximum. In case $\frac{n_1n_2}{n_{12}}$ is integral, $\frac{n_1n_2}{n_{12}}$ and $\frac{n_1n_2}{n_{12}} - 1$ both achieve the maximum.

24 The probability that John receives no mail on a given weekday can be obtained as follows:

$$p = \Pr(N = 0) = \frac{e^{-10}10^0}{0!} = e^{-10} \approx 4.54 \times 10^{-5}$$

So in 3000 days, the average number of days that John has no mail is $3000 \times 4.54 \times 10^{-5}$. Let D be the number of days that bring no mail for John in a ten-year period, then D has a Poisson distribution with mean $3000 \times 4.54 \times 10^{-5}$. Therefore, the probability John wants to find is

$$\Pr(D \ge 1) = 1 - \Pr(D = 0) = 1 - e^{-3000\times4.54\times10^{-5}} \approx 0.127\,.$$

32 Let's look at the probability mass function of $X+Y$:

$$\begin{aligned}
\Pr(X+Y=i) &= \sum_{k=0}^{i} \Pr(X=k)\Pr(Y=i-k) \\
&= \sum_{k=0}^{i} \frac{e^{-m}m^k}{k!} \cdot \frac{e^{-\bar{m}}\bar{m}^{(i-k)}}{(i-k)!} \\
&= \frac{e^{-(m+\bar{m})}}{i!} \sum_{k=0}^{i} \frac{i!}{k!(i-k)!} \cdot m^k \bar{m}^{(i-k)} \\
&= \frac{e^{-(m+\bar{m})}}{i!} \sum_{k=0}^{i} \binom{i}{k} m^k \bar{m}^{(i-k)} \\
&= \frac{e^{-(m+\bar{m})}}{i!} (m+\bar{m})^i
\end{aligned}$$

Therefore, $X+Y$ has a Poisson distribution with mean $m+\bar{m}$.

44 Consider the p.m.f. of a hypergeometric distribution

$$\begin{aligned}
h(N,k,n,x) &= \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}} = \frac{\frac{k!}{(k-x)!x!} \cdot \frac{(N-k)!}{(n-x)!(N-k-n+x)!}}{\frac{N!}{(N-n)!n!}} \\
&= \underbrace{\frac{k!(N-k)!(N-n)!}{(k-x)!(N-k-n+x)!N!}}_{\text{(a)}} \cdot \frac{n!}{x!(n-x)!}
\end{aligned}$$

Use Sterling formula $n! \approx \sqrt{2\pi n} \cdot e^{-n} n^n$ to approximate the factorial terms in part (a) and we got

$$\begin{aligned}
\text{(a)} &\approx \frac{\sqrt{k(N-k)(N-n)}}{\sqrt{(k-x)(N-k-n+x)N}} \cdot \frac{k^k (N-k)^{N-k} (N-n)^{N-n}}{(k-x)^{k-x}(N-k-n+x)^{N-k-n+x} N^N} \\
&\sim 1 \cdot \frac{k^x (N-k)^{n-x}}{N^n} \\
&= \left(\frac{k}{N}\right)^x \left(\frac{N-k}{N}\right)^{n-x} = p^x (1-p)^{n-x}
\end{aligned}$$

Hence, as k,$N \to \infty$,

$$h(N,k,n,x) \sim \binom{n}{x} p^x (1-p)^{n-x},$$

which is binomial$(n\,,p)$.