STAT 241/541 Homework 9 Solution

Section 5.2: 18, 28, 34

18 What we need is to find $F_W(w)$, where $W = aX + b$, because F_Y and F_Z can be obtained by substituting $a = 1$ and $b = 0$ into F_W, respectively.

$$\begin{aligned} F_W(w) &= \Pr(W \le w) = \Pr(aX + b \le w) \\ &= \Pr(aX \le w - b) \\ &= \begin{cases} \Pr(X \le (w-b)/a)\,, & a > 0\,; \\ \Pr(w - b \ge 0)\,, & a = 0\,; \\ \Pr(X \ge (w-b)/a)\,, & a < 0\,. \end{cases} \\ &= \begin{cases} F_X(\frac{w-b}{a})\,, & a > 0\,; \\ \Pr(w \ge b)\,, & a = 0\,; \\ 1 - F_X(\frac{w-b}{a})\,, & a < 0\,. \end{cases} \end{aligned}$$

28 Let L be the length of pregnancy, then $L \sim N(270, 10^2)$, we have

$$\begin{aligned} \Pr(\text{in the country}) &= 1 - \Pr(\text{out of the country}) \\ &= 1 - \Pr(240 < L < 290) \\ &= 1 - \Pr(-3 < \frac{L - 270}{10} < 2) \\ &= 1 - (\Phi(2) - \Phi(-3)) = 1 - (\Phi(2) - 1 + \Phi(3)) \\ &= 2 - \Phi(2) - \Phi(3) \approx 2 - 0.9772 - 0.9987 \\ &\approx 0.0241 \end{aligned}$$

34

$$\Pr(X < Y) = \int_{x=0}^{\infty} \int_{y=x}^{\infty} f(x)g(y)dxdy = \int_{x=0}^{\infty} f(x)(1 - G(x))dx\,.$$

Thus,

$$\Pr(X < Y) = \int_0^{\infty} \lambda e^{-\lambda x} \cdot e^{-\mu x} dx = \frac{\lambda}{\lambda + \mu}\,.$$

Therefore, the probability that a 100 watt bulb will outlast a 60 watt bulb is

$$\frac{1/200}{1/200 + 1/100} = \frac{1}{3}\,.$$

Section 7.2: **2, 6, 10**

2 (a) The density function of Z is given by the convolution of X and Y, i.e.,

$$f_Z(z) = \int_{-\infty}^{+\infty} f_X(z-y)f_Y(y)dy$$

For the integrand to be non-zero, we must have

$$-1 \leq z - y \leq 1 \text{ and } -1 \leq y \leq 1$$

which implies $\max(z-1,-1) \leq y \leq \min(z+1,1)$, i.e.,

$$\begin{array}{ll} \text{For} \quad -2 \leq z \leq 0\,, & -1 \leq y \leq z+1 \\ \text{For} \quad 0 < z \leq 2\,, & z-1 \leq y \leq 1 \end{array}$$

Therefore,

$$f_Z(z) = \begin{cases} \displaystyle\int_{-1}^{z+1} \frac{1}{2}\cdot\frac{1}{2}dy = \frac{z+2}{4}\,, & \text{if } -2 \leq z \leq 0 \\ \displaystyle\int_{z-1}^{1.5.6} \frac{1}{2}\cdot\frac{1}{2}dy = \frac{2-z}{4}\,, & \text{if } \ 0 < z \leq 2 \end{cases}$$

(b) With similar procedure as in part (a), we can get

$$f_Z(z) = \begin{cases} \dfrac{z+6}{4}\,, & \text{if} \quad 6 \leq z \leq 8 \\ \dfrac{10-z}{4}\,, & \text{if} \quad 8 < z \leq 10 \end{cases}$$

(c) We must have $f_X(z-y)f_Y(y)$ to be non-zero, i.e., $z-1 \leq y \leq z+1$ and $3 \leq y \leq 5$, which implies,

$$\text{if } 2 \leq z \leq 4, \text{then } 3 \leq y \leq z+1$$

$$\text{if } 4 < z \leq 6, \text{then } z-1 \leq y \leq 5$$

Thus,

$$f_Z(z) = \begin{cases} \dfrac{z-2}{4}\,, & \text{if} \quad 2 \leq z \leq 4 \\ \dfrac{6-z}{4}\,, & \text{if} \quad 4 < z \leq 6 \end{cases}$$

(d) From the answer of (a), (b) and (c), one can get the set of z such that the corresponding density functions are non-zero.

6 For this problem, one can use the proof in Harry's notes concerning $X \sim N(0,\sigma_1^2), Y \sim N(0,\sigma_2^2)$ and X, Y independent, then the density function of $Z = X + Y$ is

$$\begin{aligned} f_Z(z) &= \frac{1}{2\pi\sigma_1\sigma_2}\int \exp\Big(-\frac{y^2}{2\sigma_2^2}-\frac{(z-y)^2}{2\sigma_1^2}\Big)dy \\ &= \ldots \\ &= \frac{\sigma_1\sigma_2}{\sqrt{2\pi}\sqrt{\sigma_1^2+\sigma_2^2}}\exp\Big(-\frac{1}{2}\frac{z^2}{\sigma_1^2+\sigma_2^2}\Big) \end{aligned}$$

i.e., $Z = X + Y \sim N(0, \sigma_1^2 + \sigma_2^2)$, the details for the proof can be found in Harry's notes of lecture 25. Now, one wants to find the density function of

$$f_Z(z) = \frac{1}{2\pi\sigma_1\sigma_2}\int \exp\Big(-\frac{(y-\mu_2)^2}{2\sigma_2^2}-\frac{(z-y-\mu_1)^2}{2\sigma_1^2}\Big)dy$$

Let $y' = y - \mu_2$, $z' = z - \mu_1 - \mu_2$, then

$$f_{Z'}(z') = \frac{1}{2\pi\sigma_1\sigma_2}\int \exp\Big(-\frac{(y')^2}{2\sigma_2^2}-\frac{(z'-y')^2}{2\sigma_1^2}\Big)dy$$

which is exactly the same as the first equation and this implies

$$Z' = Z - \mu_1 - \mu_2 \sim N(0, \sigma_1^2 + \sigma_2^2)$$

Therefore, $Z' \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.

10 The c.d.f. of M is

$$\begin{aligned} \Pr(M \le m) &= \Pr\Big(\min(X_1, X_2, \ldots, X_n) \le m\Big) \\ &= 1 - \Pr\Big(\min(X_1, X_2, \ldots, X_n) \ge m\Big) \\ &= 1 - \Pr\Big(X_1 \ge m, X_2 \ge m, \ldots, X_n \ge m\Big) \\ &= 1 - \Pr(X_1 \ge m)\cdot\Pr(X_2 \ge m)\cdots\Pr(X_n \ge m) \\ &= 1 - e^{\frac{m}{\mu}}\cdot e^{\frac{m}{\mu}}\cdots e^{\frac{m}{\mu}} \\ &= 1 - e^{\frac{n}{\mu}m} \end{aligned}$$

which implies M is exponentially distributed with mean $\frac{\mu}{n}$.