1. (50 points) Suppose I have three coins in my pocket: the first lands heads with probability $0.1$, the second with probability $0.5$, and the third with probability $0.9$. I select a coin at random from my pocket and toss it twice. Let $C_i$ denote the event that I choose coin $i$, for $i = 1, 2, 3$, and $H_n$ denote the event that the $n$th toss lands heads, for $n = 1, 2$.

(a) Find $P(C_i \cap H_1)$ for $i = 1, 2, 3$.
(b) Find $P(H_1)$.
(c) Find $P(C_i|H_1)$ for $i = 1, 2, 3$.
(d) Find $P(C_i|H_1)P(H_2|C_i \cap H_1)$ for $i = 1, 2, 3$.
(e) Find $P(H_2|H_1)$.

2. (40 points) Let $X \sim U(0, 1)$.

(a) Calculate $\mathbb{E}X$ and $\mathbb{E}X^2$.

(b) Find the value of the constant $c$ for which $\mathbb{E}(X - c)^2$ is as small as possible.

(c) Find the density of $Y = - \log(X)$.

(d) Let $Y \sim $Exponential $E(1)$ with density

$$f(y) = \begin{cases} \exp(-y), & y > 0 \\ 0, & \text{otherwise} \end{cases}.$$ 

Calculate $\mathbb{E}Y$.

3. (10 points) Hat Check problem: a hat-check girl in a restaurant, having checked $n$ hats, gets them hopelessly scrambled and returns them at random to the $n$ owners as they leave. What is the expected number of people who gets his own hat back?

Formula:

$$\frac{d}{dx} \exp(-x) = -\exp(-x)$$

$$\frac{d}{dx} [(1 + x) \exp(-x)] = -x \exp(-x)$$

$$\frac{d}{dx} \frac{1}{n+1} x^{n+1} = x^n, n \neq -1$$