

Midterm Exam, 2007 Fall

Time: 9:25-10:15, Oct. 24

1. (50 points) Suppose I have three coins in my pocket: the first lands heads with probability 0.1, the second with probability 0.5, and the third with probability 0.9. I select a coin at random from my pocket and toss it twice. Let C_i denote the event that I choose coin i , for $i = 1, 2, 3$, and H_n denote the event that the n th toss lands heads, for $n = 1, 2$.
 - (a) Find $P(C_i \cap H_1)$ for $i = 1, 2, 3$.
 - (b) Find $P(H_1)$.
 - (c) Find $P(C_i|H_1)$ for $i = 1, 2, 3$.
 - (d) Find $P(C_i|H_1)P(H_2|C_i \cap H_1)$ for $i = 1, 2, 3$.
 - (e) Find $P(H_2|H_1)$.
2. (40 points) Let $X \sim U(0, 1)$.
 - (a) Calculate $\mathbb{E}X$ and $\mathbb{E}X^2$.
 - (b) Find the value of the constant c for which $\mathbb{E}(X - c)^2$ is as small as possible.
 - (c) Find the density of $Y = -\log(X)$.
 - (d) Let $Y \sim \text{Exponential } E(1)$ with density

$$f(y) = \begin{cases} \exp(-y), & y > 0 \\ 0, & \text{otherwise} \end{cases} .$$

Calculate $\mathbb{E}Y$.

3. (10 points) Hat Check problem: a hat-check girl in a restaurant, having checked n hats, gets them hopelessly scrambled and returns them at random to the n owners as they leave. What is the expected number of people who gets his own hat back?

Formula:

$$\begin{aligned} \frac{d}{dx} \exp(-x) &= -\exp(-x) \\ \frac{d}{dx} [(1+x)\exp(-x)] &= -x\exp(-x) \\ \frac{d}{dx} \frac{1}{n+1} x^{n+1} &= x^n, n \neq -1 \end{aligned}$$