Week 11
Nov. 12 – Nov. 16

Lecture 29
Definition. $X_1 \sim$Exponential(λ),

$$f_{X_1}(t) = \lambda \exp(-\lambda t), \, t > 0.$$

$X \sim$Gamma(α, λ)

$$f_X(t) = \frac{\lambda^\alpha}{\Gamma(\alpha)} t^{\alpha-1} \exp(-\lambda t), \, t > 0.$$

Theorem.

$$X \sim Gamma(\alpha_1, \lambda), \, Y \sim Gamma(\alpha_2, \lambda) \Rightarrow X + Y \sim Gamma(\alpha_1 + \alpha_2, \lambda)$$

Proof. Applying $f_{X+Y} = f_X * f_Y(y)$.
Transformation of random variables.
A quick summary table

X_i	$\sum_{i=1}^n X_i$	Remark
Bernoulli(p)	Binomial(n,p)	$Y_1 \sim$Binomial(n_1,p), $Y_2 \sim$Binomial(n_2,p), $Y_1+Y_2 \sim ?$
Geometric(p)	Negative Binomial(n,p)	$Y_1 \sim$NB(n_1,p), $Y_2 \sim$NB(n_2,p), $Y_1+Y_2 \sim ?$
Poisson(λ)	Poisson$(n\lambda)$	$Y_1 \sim$Poisson(λ_1), $Y_2 \sim$Poisson(λ_2), $Y_1+Y_2 \sim ?$
Exponential(λ)	Gamma(n,λ)	$X \sim$Gamma(α_1,λ), $Y \sim$Gamma(α_2,λ), $Y_1+Y_2 \sim ?$
Normal(μ,σ^2)	Normal$(n\mu,n\sigma^2)$	$Y_1 \sim$Normal(μ_1,σ_1^2), $Y_2 \sim$Normal(μ_2,σ_2^2), $Y_1+Y_2 \sim ?$

Transformation.
Let X be a continuous random variable with density $f_X(x)$. Let $h(x)$ be a strictly increasing function on the range of X. Define $Y = h(X)$. Then cdf is

$$F_Y(y) = F_X\left(h^{-1}(y)\right)$$

and pdf is

$$f_Y(y) = f_X\left(h^{-1}(y)\right) \cdot \left(h^{-1}(y)\right)'$$

Similarly, if Let $h(x)$ be a strictly decreasing function on the range of X. Define $Y = h(X)$. Then cdf is

$$F_Y(y) = 1 - F_X\left(h^{-1}(y)\right)$$

and pdf is

$$f_Y(y) = -f_X\left(h^{-1}(y)\right) \cdot \left(h^{-1}(y)\right)'$$

Question: Let X be a continuous random variable with cdf $F_X(x)$. Define $Y = F_X(X)$. what are cdf and pdf of Y?
All connected to uniform.

Let U denote a uniform random variable on $[0, 1]$, i.e.

$$f_U(x) = 1,\ 0 < x < 1.$$

Let X be a continuous random variable with cdf $F_X(x)$. We can built the connection of X and Z by defining

$$Y = F_X^{-1}(U).$$

Then X and Y are identically distributed!

Example (not monotone): $X \sim N(0, 1)$. Define $Y = X^2$. Find cdf and pdf of Y.

$$Y \sim \text{Gamma}(1/2, 1/2)?$$

Note that

$$\begin{aligned} F_{X^2}(t) &= P\left(X^2 \leq t\right) = P\left(-\sqrt{t} \leq X \leq \sqrt{t}\right) \\ &= 2\Phi\left(\sqrt{t}\right) - 1 \end{aligned}$$

then

$$f_{Z_1^2}(t) = \frac{1}{\sqrt{2\pi}} t^{1/2-1} \exp(-t/2).$$

Chi-square, t and Cauchy

Distribution of X/Y

Let $Z_1, Z_2, \ldots, Z_n$ be i.i.d. $N(0, 1)$.

Chi-square.

$$Y = Z_1^2 + \ldots + Z_n^2 \sim \chi_n^2$$

What is pdf of $Y = Z_1^2 + \ldots + Z_n^2$? Recall $Z_1^2 \sim$Gamma$(1/2, 1/2)$. Then

$$Y \sim Gamma(n/2, 1/2)$$

and

$$f_Y(y) = \frac{1}{\Gamma(n/2)\, 2^n} t^{n/2-1} \exp(-t/2).$$

Cauchy.

What is the distribution of $Z_1/|Z_2|$?

$$f_{Z_1/|Z_2|}(t) = \frac{1}{\pi} \frac{1}{1 + x^2}$$

Let X and Y $(Y > 0)$ be two independent random variables, and f_X and f_Y be the densities of X and Y respectively. Let $Z = X/Y$. What is distribution of Z? For example,

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2},\ f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}.$$

The joint density of (X, Y) is

$$f_{X,Y}(x,y) = f_X(x)\, f_Y(y).$$

Then

$$\begin{aligned} P(X/Y \le t) &= P(0 < Y < \infty, X/Y \le t) \\ &= \int_0^\infty \int_{-\infty}^{ty} f_X(x)\, f_Y(y)\, dxdy \\ &= \int_0^\infty \left[\int_{-\infty}^{ty} f_X(x)\, dx \right] f_Y(y)\, dy. \end{aligned}$$

Let $G(t) = \int_{-\infty}^{ty} f_X(x)\, dx$. We write

$$F_{X/Y}(t) = P(X/Y \le t) = \int_0^\infty G(t)\, f_Y(y)\, dy.$$

This implies

$$F'_{X/Y}(t) = \int_{-\infty}^\infty G'(t)\, f_Y(y)\, dy$$

i.e.,

$$f_{X/Y}(t) = \int_{-\infty}^\infty y f_X(ty)\, f_Y(y)\, dy$$

We see

$$\begin{aligned} f_{Z_1}(x) &= \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \\ f_{|Z_2|}(y) &= \frac{2}{\sqrt{2\pi}} e^{-y^2/2} \end{aligned}$$

Then

$$\begin{aligned} f_{Z_1/|Z_2|}(t) &= \frac{1}{\pi} \int_0^\infty y \exp\left(-\frac{(ty)^2}{2} - \frac{y^2}{2} \right) dy \\ &= \frac{1}{\pi} \int_0^\infty \left[-\frac{1}{1+t^2} \exp\left(-\frac{(ty)^2}{2} - \frac{y^2}{2} \right) \right]' dy = \frac{1}{\pi} \frac{1}{1+t^2}. \end{aligned}$$

Students' t distribution.

$$T = \frac{Z_1}{\sqrt{\frac{Z_2^2 + \ldots + Z_n^2}{n-1}}} \sim t_{n-1}$$

Similarly we have

$$f_T(t) = \frac{\Gamma(n/2)}{\sqrt{(n-1)\pi}\,\Gamma\left(\frac{n-1}{2}\right)} \left(1 + \frac{t^2}{n-1} \right)^{-n/2}.$$

Lecture 30. Law of Large Number
Hint for one homework problem
Cauchy.
What is the distribution of $Z_1/|Z_2|$?

$$f_{Z_1/|Z_2|}(t) = \frac{1}{\pi}\frac{1}{1+x^2}$$

Let X and Y $(Y > 0)$ be two independent random variables, and f_X and f_Y be the densities of X and Y respectively. Let $Z = X/Y$. What is distribution of Z? For example,

$$f_X(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2},\ f_Y(y) = \frac{1}{\sqrt{2\pi}}e^{-y^2/2}.$$

The joint density of (X, Y) is

$$f_{X,Y}(x,y) = f_X(x) f_Y(y).$$

Then

$$\begin{aligned} P(X/Y \le t) &= P(0 < Y < \infty, X/Y \le t) \\ &= \int_0^\infty \int_{-\infty}^{ty} f_X(x) f_Y(y)\, dx dy = \int_0^\infty \left[\int_{-\infty}^{ty} f_X(x)\, dx\right] f_Y(y)\, dy. \end{aligned}$$

Let $G(t) = \int_{-\infty}^{ty} f_X(x)\, dx$. We write

$$F_{X/Y}(t) = P(X/Y \le t) = \int_0^\infty G(t) f_Y(y)\, dy.$$

This implies

$$F'_{X/Y}(t) = \int_{-\infty}^\infty G'(t) f_Y(y)\, dy$$

i.e.,

$$f_{X/Y}(t) = \int_{-\infty}^\infty y f_X(ty) f_Y(y)\, dy$$

We see

$$f_{Z_1}(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2},\ f_{|Z_2|}(y) = \frac{2}{\sqrt{2\pi}}e^{-y^2/2}$$

Then

$$f_{Z_1/|Z_2|}(t) = \frac{2}{\pi}\int_0^\infty y \exp\left(-\frac{(ty)^2}{2} - \frac{y^2}{2}\right) dy = \frac{1}{\pi}\frac{1}{1+t^2}.$$

Law of Large Number

Example: Flip a coin $n = 2000$ times. Assume that the chance to get a head is p in each flip. Let S be the total number of heads. You observe 1200 heads. Are you confident to say that $p > 1/2$?

Question: Define a new random variable $X = \frac{S}{n}$.

$$\begin{aligned} EX &= ? \\ Var(X) &= ?. \end{aligned}$$

Is it true that

$$Var(X) = \frac{p(1-p)}{2000} \leq \frac{1}{8000}?$$

Question: Our possible estimate of p would be $1200/2000 = 0.6$. Let $A = \{X : |X - p| \geq 0.1\}$.

$$E(|X - p| \geq 0.1) = ?$$

A satisfactory answer to this question: Define

$$Y = \begin{cases} |X - p|^2 & \text{if } |X - p| \geq 0.1 \\ 0 & otherwise \end{cases}$$

It must be true that

$$EY \leq E|X - p|^2 = \frac{p(1-p)}{2000}$$

Since

$$Y \leq |X - p|^2.$$

Define

$$Z = \begin{cases} 0.01 & \text{if } |X - p| \geq 0.1 \\ 0 & otherwise \end{cases}$$

then

$$0.1^2 P(|X - p| \geq 0.1) \leq EY$$

since

$$Z \leq Y.$$

Thus we have

$$0.1^2 P(|X - p| \geq 0.1) \leq \frac{p(1-p)}{2000} \leq \frac{1/4}{2000}$$

i.e.,

$$P(|X - p| \geq 0.1) \leq 1/80 = 1.25\%.$$

Theorem: Let X be a random variable with $EX = \mu$ and $Var(X) = \sigma^2$. Let $\epsilon > 0$ be any positive real number. Then

$$P(|X - \mu| \geq \epsilon) \leq \frac{\sigma^2}{\epsilon^2}$$

Proof: Define

$$Y = \begin{cases} |X-\mu|^2 & \text{if } |X-\mu| \geq \epsilon \\ 0 & otherwise \end{cases}$$

and

$$Z = \begin{cases} \epsilon^2 & \text{if } |X-\mu| \geq \epsilon \\ 0 & otherwise \end{cases} .$$

We see

$$Z \leq Y \leq |X-\mu|^2 = \sigma^2.$$

Then

$$\epsilon^2 P\left(|X-\mu| \geq \epsilon\right) \leq EZ \leq EY \leq E\,|X-\mu|^2 = \sigma^2.$$

Lecture 31. Central Limit Theorem

Theorem: Let $X_1, X_2, \ldots, X_n$ be i.i.d. with $EX_i = \mu$ and $Var(X_i) = \sigma^2$. Let $\overline{X} = \frac{X_1+X_2+\ldots+X_n}{n}$. Then the distribution of $\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}$ is approximately $N(0,1)$.

Example: Flip a coin $n = 2000$ times. Assume that the chance to get a head is p in each flip. Let S be the total number of heads. You observe 1200 heads. Are you confident to say that $p > 1/2$?

No. $P\left(\left|\frac{S}{n} - p\right| \geq 0.1\right) \leq 1/80 = 1.25\%$.

A more delicate analysis: the distribution of $\frac{S}{n} - p$ is approximately

$$N\left(0, \frac{p(1-p)}{n}\right)$$

i.e., the distribution of $\frac{\frac{S}{n}-p}{\sqrt{\frac{p(1-p)}{n}}}$ is approximately

$$N(0,1)$$

We know

$$\frac{0.1}{\sqrt{\frac{p(1-p)}{n}}} \geq \frac{0.1}{\sqrt{\frac{1}{8000}}} \approx 8.9.$$

and

$$P\left(\left|\frac{S}{n} - p\right| \geq 0.1\right) \approx 2(1 - \Phi(8.9)) \leq 10^{-16}.$$

Today we will show that: Let $X_1, X_2, \ldots, X_n$ be i.i.d. Bernoulli(p) with $n = 2k$ and $p = 1/2$, then $\frac{\overline{X}-p}{\sqrt{p(p-1)/n}}$ is approximately $N(0,1)$.

Let $X = X_1 + X_2 + \ldots + X_n \sim Binomial(n, p)$.

Question:

$$P(X = n/2) \sim \sqrt{\frac{1}{\pi k}} = \sqrt{\frac{2}{\pi n}}?$$

(It was shown in the topic of random walk).

Now we want to show

$$P(X = n/2 + m) \sim \sqrt{\frac{2}{\pi n}} \exp\left(-\frac{m^2}{n/2}\right) = \frac{1}{\sqrt{2\pi}\cdot\sqrt{n/4}} \sqrt{\frac{2}{\pi n}} \exp\left(-\frac{m^2}{2\cdot n/4}\right)$$

Recall that

$$\frac{P(X = i+1)}{P(X = i)} = \frac{\binom{n}{i+1}}{\binom{n}{i}} = \frac{n-i}{i+1}$$

Then

$$\begin{aligned}
\frac{P(X = k+1)}{P(X = k)} &= \frac{n-k}{k+1} \approx 1 \\
&\cdots \\
\frac{P(X = k+m)}{P(X = k+m-1)} &= \frac{n-k-m+1}{k+m-1} = \frac{1 - \frac{m-1}{k}}{1 + \frac{m-1}{k}} \approx 1 - \frac{m-1}{2k}
\end{aligned}$$

where $k = n/2$. Note that

$$\log(1-x) \approx x, \text{ when } x \text{ is small.}$$

We then have

$$\begin{aligned}
& \log \frac{P(X=k+m)}{P(X=k)} \\
= \quad & \log \frac{P(X=k+1)}{P(X=k)} + \log \frac{P(X=k+1)}{P(X=k)} + \ldots + \log \frac{P(X=k+m)}{P(X=k+m-1)} \\
\approx \quad & -\frac{1}{2k} + \ldots + \frac{m-1}{2k} \\
\approx \quad & -\frac{m^2}{4k} = -\frac{m^2}{n/2}.
\end{aligned}$$

Thus

$$P(X = n/2 + m) \sim \sqrt{\frac{2}{\pi n}} \exp\left(-\frac{m^2}{n/2}\right)$$

Lecture 32. Moment Generating function

Studying moment generating function will help us to understand *Central limit theorem* for general cases!

Let $X \sim Binomial\,(n,p)$. We may ask a question

$$\begin{aligned} EX^3 &= ? \\ EX^4 &= ? \end{aligned}$$

A similar question can be asked for Poisson, Negative Binomial, Gamma and Normal, etc.

A calculus trick: if we know how to calculate Ee^{tX} and then set

$$g(t) = Ee^{tX}$$

then

$$\begin{aligned} g'(0) &= EX \\ g''(0) &= EX^2 \\ &\cdots \\ g^{(k)}(0) &= EX^k \end{aligned}$$

Example: For $X \sim Binomial\,(n,p)$, it is easy to see

$$g_X(t) = \left(1 - p + pe^t\right)^n.$$

Why? There are at least two ways to answer this question!

This formula implies

$$\begin{aligned} EX &= npe^t\left(1 - p + pe^t\right)^{n-1}|_{t=0} = np \\ EX^2 &= np + n(n-1)p^2 = np(1-p) \end{aligned}$$

Example: For $X \sim Poisson\,(\lambda)$, it can be shown

$$g_X(t) = e^{\lambda\left(e^t - 1\right)}.$$

This formula implies

$$\begin{aligned} EX &= \lambda e^t e^{\lambda\left(e^t-1\right)}|_{t=0} = \lambda \\ EX^2 &= \lambda + \lambda^2 \end{aligned}$$

Simple but important properties of moment generating function

(i) Let X and Y be independent

$$g_{X+Y}(t) = g_X(t)\,g_Y(t).$$

(ii) Let $Y = \frac{X-\mu}{\sigma}$, then

$$g_Y(t) = e^{-\mu t/\sigma} g_X\left(\frac{t}{\sigma}\right).$$