Week 12

Nov. 26 – Nov. 30

Lecture 32. Central Limit Theorem

Theorem: Let X_1, X_2, $\ldots$, X_n be i.i.d. with $EX_i = \mu$ and $Var\left(X_i\right) = \sigma^2$. Let $\overline{X} = \frac{X_1+X_2+\ldots+X_n}{n}$. Then the distribution of $\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}$ is approximately $N\left(0,1\right)$.

Example: Flip a coin $n = 2000$ times. Assume that the chance to get a head is p in each flip. Let S be the total number of heads. You observe 1200 heads. Are you confident to say that $p > 1/2$?

No. $P\left(\left|\frac{S}{n}-p\right| \geq 0.1\right) \leq 1/80 = 1.25\%$.

A more delicate analysis: the distribution of $\frac{S}{n} - p$ is approximately

$$N\left(0, \frac{p\left(1-p\right)}{n}\right)$$

i.e., the distribution of $\frac{\frac{S}{n}-p}{\sqrt{\frac{p(1-p)}{n}}}$ is approximately

$$N\left(0,1\right)$$

We know

$$\frac{0.1}{\sqrt{\frac{p(1-p)}{n}}} \geq \frac{0.1}{\sqrt{\frac{1}{8000}}} \approx 8.9.$$

and

$$P\left(\left|\frac{S}{n}-p\right| \geq 0.1\right) \approx 2\left(1-\Phi\left(8.9\right)\right) \approx 0.$$

Today we will show that: Let X_1, X_2, $\ldots$, X_n be i.i.d. Bernoulli(p) with $n = 2k$ and $p = 1/2$, then $\frac{\overline{X}-p}{\sqrt{p(p-1)/n}}$ is approximately $N\left(0,1\right)$.

Let $X = X_1 + X_2 + \ldots + X_n \sim \mathit{Binomial}\left(n,p\right)$.

Question:

$$P\left(X = n/2\right) \sim \sqrt{\frac{1}{\pi k}} = \sqrt{\frac{2}{\pi n}}?$$

(It was shown in the topic of random walk).

Now we want to show

$$P\left(X = n/2+m\right) \sim \sqrt{\frac{2}{\pi n}} \exp\left(-\frac{m^2}{n/2}\right) = \frac{1}{\sqrt{2\pi}\cdot\sqrt{n/4}}\sqrt{\frac{2}{\pi n}} \exp\left(-\frac{m^2}{2\cdot n/4}\right)$$

Recall that

$$\frac{P\left(X=i+1\right)}{P\left(X=i\right)} = \frac{\binom{n}{i+1}}{\binom{n}{i}} = \frac{n-i}{i+1}$$

Then

$$\begin{aligned}
\frac{P\left(X=k+1\right)}{P\left(X=k\right)} &= \frac{n-k}{k+1} \approx 1 \\
&\dots \\
\frac{P\left(X=k+m\right)}{P\left(X=k+m-1\right)} &= \frac{n-k-m+1}{k+m-1} = \frac{1-\frac{m-1}{k}}{1+\frac{m-1}{k}} \approx 1-\frac{m-1}{2k}
\end{aligned}$$

where $k = n/2$. Note that

$$\log\left(1-x\right) \approx x, \text{ when } x \text{ is small.}$$

We then have

$$\begin{aligned}
& \log\frac{P\left(X=k+m\right)}{P\left(X=k\right)} \\
= \;& \log\frac{P\left(X=k+1\right)}{P\left(X=k\right)} + \log\frac{P\left(X=k+1\right)}{P\left(X=k\right)} + \ldots + \log\frac{P\left(X=k+m\right)}{P\left(X=k+m-1\right)} \\
\approx \;& -\frac{1}{2k} + \ldots + \frac{m-1}{2k} \\
\approx \;& -\frac{m^2}{4k} = -\frac{m^2}{n/2}.
\end{aligned}$$

Thus

$$P\left(X=n/2+m\right) \sim \sqrt{\frac{2}{\pi n}}\exp\left(-\frac{m^2}{n/2}\right)$$

Lecture 33. Moment Generating function (I)

Central Limit Theorem

Theorem: Let X_1, X_2, $\ldots$, X_n be i.i.d. with $EX_i = \mu$ and $Var\,(X_i) = \sigma^2$. Let $\overline{X} = \frac{X_1+X_2+\ldots+X_n}{n}$. Then the distribution of $\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}$ is approximately $N\,(0,1)$ in the sense that

$$\lim_{n\to\infty} P\left(a \le \frac{\overline{X}-\mu}{\sigma/\sqrt{n}} \le b\right) = \Phi\,(b) - \Phi\,(a) \text{ for all } a, b.$$

Studying moment generating function will help us to understand *Central limit theorem* for general cases!

Let $X \sim Binomial\,(n,p)$. We may ask a question

$$\begin{aligned} EX^3 &= ? \\ EX^4 &= ? \end{aligned}$$

A similar question can be asked for Poisson, Negative Binomial, Gamma and Normal, etc.

A calculus trick: if we know how to calculate Ee^{tX} and then set

$$g\,(t) = Ee^{tX}$$

then

$$\begin{aligned} g'\,(0) &= EX \\ g''\,(0) &= EX^2 \\ &\cdots \\ g^{(k)}\,(0) &= EX^k \end{aligned}$$

Example: For $X \sim Binomial\,(n,p)$, it is easy to see

$$g_X\,(t) = \left(1-p+pe^t\right)^n.$$

Why? There are at least two ways to answer this question!

This formula implies

$$\begin{aligned} EX &= npe^t\left(1-p+pe^t\right)^{n-1}|_{t=0} = np \\ EX^2 &= np + n\,(n-1)\,p^2 \\ Var\,(X) &= np\,(1-p) \end{aligned}$$

Example: For $X \sim Poisson\,(\lambda)$, it can be shown

$$\begin{aligned} g_X\,(t) &= \sum_{k=0}^{\infty} e^{tk}\frac{\lambda^k}{k!}e^{-\lambda} \\ &= e^{-\lambda}\sum_{k=0}^{\infty}\frac{\left(\lambda e^t\right)^k}{k!} \\ &= e^{-\lambda}e^{\lambda e^t} = e^{\lambda\left(e^t-1\right)} \end{aligned}$$

This formula implies

$$\begin{aligned} EX &= \lambda e^t e^{\lambda\left(e^t-1\right)}|_{t=0} = \lambda \\ EX^2 &= \lambda + \lambda^2 \\ Var\left(X\right) &= \lambda \end{aligned}$$

Simple but important properties of moment generating function
(i) Let X and Y be independent

$$g_{X+Y}\left(t\right) = g_X\left(t\right) g_Y\left(t\right).$$

(ii) Let $Y = \frac{X-\mu}{\sigma}$, then

$$g_Y\left(t\right) = e^{-\mu t/\sigma} g_X\left(\frac{t}{\sigma}\right).$$

Lecture 34. Moment Generating function (II)

Fact: If two random variables have exactly the same moment generating functions, then their cumulative distribution functions are identical!

Sorry. We are not proving this fact, but using it.

Example: For $X \sim Gamma\,(\alpha, \lambda)$, it is easy to see

$$\begin{aligned} g_X(t) &= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty e^{tx} x^{\alpha-1} e^{-\lambda x} dx \\ &= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{-(\lambda-t)x} dx \\ &= \frac{\lambda^\alpha}{(\lambda-t)^\alpha} \end{aligned}$$

Remark: This calculation suggests another way to do Problem 1 in Homework 10.

Problem 1. Let $X \sim Gamma\,(\alpha_1, \lambda)$ and $Y \sim Gamma\,(\alpha_2, \lambda)$ be independent. Show that

$$X + Y \sim Gamma\,(\alpha_1 + \alpha_2, \lambda)$$

Solution: the moment generating function of $X + Y$ is

$$g_{X+Y}(t) = g_X(t)\, g_Y(t) = \frac{\lambda^{\alpha_1+\alpha_2}}{(\lambda-t)^{\alpha_1+\alpha_2}}$$

which is exactly the moment generating function of $Gamma\,(\alpha_1 + \alpha_2, \lambda)$.

Remark:

$$\begin{aligned} EX &= g_X'(t)\,|_{t=0} = \frac{\alpha}{\lambda} \\ EX^2 &= g_X''(t)\,|_{t=0} = \frac{\alpha(\alpha+1)}{\lambda^2} \\ Var(X) &= \frac{\alpha}{\lambda^2} \end{aligned}$$

Example: For $X \sim Normal\left(\mu, \sigma^2\right)$, we have

$$g_X(t) = \int_{-\infty}^\infty e^{tx} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx.$$

Let $y = \frac{x-\mu}{\sigma}$, then

$$\begin{aligned} g_X(t) &= \int_{-\infty}^\infty e^{t(\mu+\sigma y)} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \\ &= e^{t\mu + \left(\sigma^2/2\right)t^2} \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-\sigma t)^2}{2}} dy = e^{t\mu + \left(\sigma^2/2\right)t^2}. \end{aligned}$$

Remark: This calculation suggests another way to do Problem 6 in Chapter 7.

Problem 6 in Chapter 7. Let $X \sim Normal\left(\mu_1, \sigma_1^2\right)$ and $Y \sim Normal\left(\mu_2, \sigma_2^2\right)$ be independent. Show that

$$X + Y \sim Normal\left(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2\right)$$

Solution: the moment generating function of $X + Y$ is

$$g_{X+Y}\left(t\right) = g_X\left(t\right) g_Y\left(t\right) = e^{t(\mu_1+\mu_2)+\left(\sigma_1^2+\sigma_2^2\right)t^2/2}$$

which is exactly the moment generating function of $Normal\left(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2\right)$.

Lecture 35. Central Limit Theorem

Theorem: Let X_1, X_2, $\ldots$, X_n be i.i.d. with $EX_i = \mu$ and $Var\left(X_i\right) = \sigma^2$. Let $\overline{X} = \frac{X_1+X_2+\ldots+X_n}{n}$. Then the distribution of $\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}$ is approximately $N\left(0,1\right)$ in the sense that

$$\lim_{n\to\infty} P\left(a \leq \frac{\overline{X}-\mu}{\sigma/\sqrt{n}} \leq b\right) = \Phi\left(b\right) - \Phi\left(a\right) \text{ for all } a, b.$$

Fact: If two random variables have close moment generating functions, then their cumulative distribution functions are close!

Sorry. We are not proving this fact, but using it.

Without loss of generality we assume that $\mu = 0$ and $\sigma = 1$. Let $g\left(t\right)$ be the moment generating function of X_i. The moment generating function of $X_1 + X_2 + \ldots + X_n$ is

$$\left(g\left(t\right)\right)^n$$

then the moment generating function of $\frac{\overline{X}-\mu}{\sigma/\sqrt{n}} = \frac{X_1+X_2+\ldots+X_n-n\mu}{\sqrt{n}\sigma} = \frac{X_1+X_2+\ldots+X_n}{\sqrt{n}}$ (for $\mu = 0$ and $\sigma = 1$) is

$$g\left(t\right) = \left(g\left(\frac{t}{\sqrt{n}}\right)\right)^n$$

(**Recall:** Let $Y = \frac{X-\mu}{\sigma}$, then $g_Y\left(t\right) = e^{-\mu t/\sigma} g_X\left(\frac{t}{\sigma}\right)$.)

Remark:

$$g\left(0\right) = 1,\ g'\left(0\right) = 0,\ g''\left(0\right) = 1.$$

This remark implies

$$g\left(\frac{t}{\sqrt{n}}\right) \approx 1 + \frac{t^2}{2n}$$

then

$$g\left(t\right) = \left(g\left(\frac{t}{\sqrt{n}}\right)\right)^n \approx \left(1 + \frac{t^2}{2n}\right)^n \to e^{t^2/2} \text{ as } n \to \infty$$

and $e^{t^2/2}$ is exactly the moment generating function of $Normal\left(0,1\right)$.

Remark: If the moment generating function of X_i doesn't exist, our calculation here doesn't make sense.

Example:For $X \sim Cauchy\left(0,1\right)$, i.e.,

$$f_X\left(x\right) = \frac{1}{\pi}\frac{1}{1+x^2}$$

it doesn't have a moment generating function, since

$$\int_{-\infty}^{\infty} e^{tx}\frac{1}{\pi}\frac{1}{1+x^2}dx$$

doesn't exist!

A fun fact: Let X_1, X_2, $\ldots$, X_n be i.i.d. $Cauchy\left(0,1\right)$, then

$$\overline{X} = \frac{X_1+X_2+\ldots+X_n}{n} \sim Cauchy\left(0,1\right).$$

then

$$\lim_{n\to\infty} P\left(a \leq \frac{\overline{X}}{\sqrt{n}} \leq b\right) = \lim_{n\to\infty} P\left(\sqrt{n}a \leq \overline{X} \leq b\sqrt{n}\right) = 1.$$