STAT 241/541, Probability Theory with Applications
Fall 2012

Instructor:
Harrison H. Zhou (huibin.zhou@yale.edu)
Office hours: Thursday 4:00-6:00pm (tentative) or by appointments, Room 204, 24 Hillhouse Ave., James Dwight Dana House.

T.A.:
Zhao Ren (zhao.ren@yale.edu)
Office hours: TBA
David Brinda (william.brinda@yale.edu)
Office hours: TBA
TA session: Thursday 6:30pm-7:30pm (tentative). Homework discussion. We meet in 24 Hillhouse Avenue, Room 107. Optional, but recommended.

Class Time and Location:
MWF 9:25-10:15am.
WLH 208 (William L. Harkness Hall, 100 Wall St.).

Grade:
Weekly Homework: 25%. **Due very Friday in class.**
Midterm: 20%.
Final Exam: 45%.
Participation: 10%

Textbook:
Introduction to Probability by Charles M. Grinstead and J. Laurie Snell.

Course Homepage:
http://www.stat.yale.edu/~hz68/241/

Lectures will cover the following topics:

- Discrete probability distributions: uniform and general. Random variables. Expectation.
- Continuous probability densities: uniform and general. Buffon's needle. Cumulative distributions functions and transformation. Expectation.
- Discrete conditional probability and paradoxes. Independence. Bayes's formula and the prisoner's dilemma. Gambler's Ruin.
- Continuous conditional distribution. Independence. Combinatorics and Binomial distribution. Always coming back.

- Inclusion and Exclusion Principle and hat check problem. Important discrete distributions. Important continuous densities.
- Sum of independent random variables: Binomial and Poisson. Expectation and Variance. The multinomial. The multinomial from Poisson. Covariance and correlation.
- Sum of independent random variables: uniform,exponential, normal, and Cauchy. Expectation and Variance. Bivariate normal. Covariance and correlation.
- Transformations in many distributions.
- Law of large numbers. Strong Law.
- Central Limit Theorem. Binomial.
- Generating Functions. General CLT.
- Markov Chain.

Week 1

Lecture 1

Dice and Points.

The history of probability begins with a letter from Pascal to Fermat, on 29 July 1654, in which he solves two problems posed to him by gambler Mr. Chevalier de Méré.

PROBLEM OF DICE: We undertake to throw a six in four tosses of a die, or to throw two sixes with two dice in twenty-four throws. Which is more probable?

PROBLEMS OF POINTS: Two players each stake 32 pistoles on a game of three points. The players have equal chances of winning each point, and the first to win 3 points win the other's stake.

Suppose that the first player has gained two points and the second player has gained one point, but the players separate without finishing the game. How much should the first player receive from the second.

Some examples to be considered in this course:

1. **Buffon's needle.** In mathematics, Buffon's needle problem is a question first posed in the 18th century by Georges-Louis Leclerc, Comte de Buffon: suppose we have a floor made of parallel strips of wood, each the same width 1, and we drop a needle with length $l = 1$ onto the floor. What is the probability that the needle will lie across a line between two strips? This problem can be solved to approximate π.

2. **Rare disease.** A doctor gives a patient a test for a particular cancer. Before the results of the test, the only evidence the doctor has to go on is that 1 woman in 1000 has this cancer. Experience has shown that, in 99% of the cases in which cancer is present, the test is positive; and in 95% of the cases in which cancer is not present, it is negative. If the test turns out to be positive, what probability should the doctor assign to the event that cancer is present?

3. **The prisoner's dilemma.** Three prisoners, A, B, and C, with apparently equally good records have applied for parole. The parole board has decided to release two of the three, and the prisoners know this but not which two. A warder friend of prisoner A knows who are to be released. Prisoner A realizes that it would be unethical to ask the warder if he, A, is to be released, but thinks of asking for the name of one prisoner other than himself who is to be released. He thinks that before he asks, his chances of release are 2/3. He thinks that if the warder says "B will be released," his own chances have now gone down to 1/2, because either A and B or B and C are to be released. And so A decides not to reduce his chances by asking. However, A is mistaken in his calculations. Explain.

4. **Random Walk.** Imagine now a drunkard walking randomly in an idealized city. The city is effectively infinite and arranged in a square grid, and at every intersection, the drunkard chooses one of the four possible routes (including the one he came from) with equal probability. Formally, this is a random walk on the set of all points in the plane with integer coordinates. Will the drunkard ever get back to his home from the bar? It turns out that he will.

5. **Gambler's ruin.** You enter a casino with \$1 in your pocket. You stake \$$a$. Your return is \$$2a$ with probability $p > 1/2$; you lose \$$a$ with probability $1 - p < 1/2$. If the casino lets you play long enough and allows you to stake arbitrarily small amount, can you eventually own the casino?

6. **Hat Check problem**. A hat-check girl in a restaurant, having checked n hats, gets them hopelessly scrambled and returns them at random to the n owners as they leave. What is the probability that nobody gets his own hat back?

7. **Poisson distribution.** Consider the statistics of flying bomb hits in the south of London during World War II. The entire area is divided into a grid of $N = 576$ small areas of size one quarter square kilometer each. The table below records the number of squares with $0, 1, 2, 3$ etc. hits each. The total number of hits is 537. The average number of hits per square is then $537/576 = .93$ hits per square. It can be shown that if the targeting is completely random, then the probability that a square is hit with $0, 1, 2, 3$ etc. hits is governed by a Poisson distribution.

Number of hits k	0	1	2	3	4	5 or more
Number of areas, k hits	229	211	93	35	7	1
$576 \cdot P(X = k)$, $X \sim Po(.93)$	227	211	99	31	7	2

8. **Election.** For a survey of 2000 people in Connecticut, you find 1040 people are planning to vote for Obama. Are you confident to say that Obama is winning Connecticut?

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Homework 1:
Chapter 1.2 : 14, 18, 20, 22, 26, 28. Due Sept. 7.

Lecture 2.

Example: Consider an experiment in which a coin is tossed two times. Two possible sample space $\Omega_1 = \{HH, HT, TH, TT\}$ and $\Omega_2 = \{2\ Hs, 1\ H, 0\ H\} = \{\omega_2, \omega_1, \omega_0\}$.

Question: what is the probability that a family of two children has (i) two boys given that the first child is a boy? (ii) two boys given that it has at least one boy?

Definition: Let Ω be a finite sample space. A probability measure is a function P that assigns a number to each event that satisfies the following properties:

1. $\mathbb{P}(\{\omega\}) \geq 0$ for all $\omega \in \Omega$.
2. $\sum_{\omega\in\Omega} \mathbb{P}(\{\omega\}) = 1$.
3. $\mathbb{P}(E) = \sum_{\omega\in E} \mathbb{P}(\{\omega\})$.

Venn Diagrams. Probability axioms. Random variable. Expectation.

Notations

Name	Notation	Description
complement	E^c	occurs if and only if E doesn't occur
union	$E \cup F$	occurs when either E or F occurs
intersection	$E \cap F$	occurs when both E and F occurs
implication	$E \subset F$	F occurs whenever E occurs.

Venn Diagrams:

$$\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F)$$

and

$$(E \cup F)^c = E^c \cap F^c.$$

Probability axioms. A probability measure is a function P that assigns a number to each event that satisfies the following properties:

1. $\mathbb{P}(E) \geq 0$ for all $E \subset \Omega$.
2. $\mathbb{P}(\Omega) = 1$.
3. Let E_i be pairwise disjoint subsets of Ω, then $\mathbb{P}(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mathbb{P}(E_i)$.

Random variable. A random variable is a function defined on the sample space.

Expectation. Let X be a numerically-valued discrete random variable with sample space Ω. The *expected value* $\mathbb{E}X$ (or $\mathbb{P}X$) is defined by

$$\mathbb{E}X = \sum_{x\in\Omega} \mathbb{P}(X = x)\, x.$$