Week 2

Lecture 3. Expectation. Sample spaces with infinite outcomes

Review:

Probability axioms. A probability measure is a function $\mathbb{P}$ that assigns a number to each event that satisfies the following properties:

1. $\mathbb{P}(E) \geq 0$ for all $E \subset \Omega$.
2. $\mathbb{P}(\Omega) = 1$.
3. Let E_i be pairwise disjoint subsets of Ω, then $\mathbb{P}(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mathbb{P}(E_i)$.

Random variable. A random variable is a function defined on the sample space.

Example: Flip a coin three times, then the sample space is

$$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

Let X to be the number of heads of the outcome.

$$X = \begin{cases} 0 & 1/8 \\ 1 & 3/8 \\ 2 & 3/8 \\ 3 & 1/8 \end{cases}.$$

What is the average (expected) number of heads you will see?

The beginning of expectation: PROBLEMS OF POINTS. Two players each stake 32 pistoles on a game of three points. The players have equal chances of winning each point, and the first to win 3 points win the other's stake. Suppose that the first player has gained two points and the second player has gained one point, but the players separate without finishing the game. How much should the first player receive from the second.

Let X be a numerically-valued discrete random variable with sample space Ω. The **expected value** $\mathbb{E}X$ is defined by

$$\mathbb{E}X = \sum_{x \in \Omega} \mathbb{P}(X = x)\, x.$$

We use the following sample space

$$\Omega = \{3, 2, 1, 0\}$$

where i means i heads.

Question: Let Y be the number of the square of the number of heads of the outcome. What is $\mathbb{P}Y$?

Question: Define a random variable

$$X = \begin{cases} 1 & x \in E \\ 0 & \text{otherwise} \end{cases}$$

What is $\mathbb{E}X$? We often denote X by $\{E\}$ or $I_E(x)$.

Example 1: Flip a coin until you see a head. The outcome is the number of flips. $\Omega = \{1, 2, 3, \ldots\}$.

Question:

$$\mathbb{P}(\{i\}) = ?$$

Let

$$X(i) = i.$$

What is the expectation of X?

Example 2: Draw a number ω from 0 to 1. Every number has equal chance to be drawn.

Then the sample space is $[0, 1]$. Define $X(\omega) = \omega$. Then X is a random variable.

Question: For $E \subset [0, 1] = \Omega$, then what is the definition of $\mathbb{P}(E)$?

Question: Are the probability axioms satisfied?

Question: What is the expected value of X? Understand it from a view of discrete approximation:

$$\mathbb{E}X \approx \sum_{i=1}^{n} \frac{i}{n} \cdot \frac{1}{n} \approx ?$$

Question: What is the expected value of X^2? Why?

Question: Generalized this problem to an interval $[a, b]$. Then for $E \subset \Omega$, what is $\mathbb{P}(E)$?

Question: A real example for uniform distribution.

Definition: The cumulative distribution function of a random variable is defined as

$$F_X(x) = \mathbb{P}(X \leq x).$$

Example 3: Draw a number ω from 0 to 1. Every number on $[0, 1/2)$ has equal chance to be drawn, and so is every number on $[1/2, 1]$. The chance to draw a number from $[1/2, 1]$ doubles the chance to draw a number from $[0, 1/2)$. Let $E \subset \Omega$. What is $\mathbb{P}(E)$?

Definition: Let $f(x) \geq 0$ and $\int f(x)\,dx = 1$. (We call it probability density function). Define $\mathbb{P}(E)$ as following

$$\mathbb{P}(X \in E) = \int_E f(x)\,dx.$$

Question: Are the probability axioms satisfied?

Question: How to calculate the cdf?

Question: How to define $\mathbb{E}X$? How to define $\mathbb{E}g(X)$?

Lecture 4. Examples of continuous distribution functions. Transformation.

Review:

The cumulative distribution function . $F_X(x) = \mathbb{P}(X \le x)$

Let the probability density function be f, then

$$F_X(a) = \mathbb{P}(X \le a) = \int_{-\infty}^{a} f(x)\,dx.$$

Question: if f is continuous at a, then

$$\frac{dF_X(x)}{dx}|_{x=a} = f(a)\,?$$

Expectation:

$$\mathbb{E}g(X) = \int g(x)\,f(x)\,dx.$$

Example 1: $X \sim$Uniform$[0,1]$. What are pdf, cdf, $\mathbb{E}X$ and $\mathbb{E}X^2$?

Example 2: $X \sim$Exponential $E(1)$. What are pdf, cdf, $\mathbb{E}X$ and $\mathbb{E}X^2$?

Example 3: Cauchy distribution. What are pdf, cdf, $\mathbb{E}X$ and $\mathbb{E}X^2$?

Transformation.

Question: Let $X \sim$Uniform$[0,1]$. What are pdf and cdf of $2X$? What are pdf and cdf of X^2?

Question: Let $X \sim$Exponential(1). What are pdf and cdf of $2X$? What are pdf and cdf of X^2?

Lecture 5. Buffon's needle and Related problems

Buffon's needle.

In mathematics, Buffon's needle problem is a question first posed in the 18th century by Georges-Louis Leclerc, Comte de Buffon: suppose we have a floor made of parallel strips of wood, each the same width 1, and we drop a needle with length $l = 1$ onto the floor. What is the probability that the needle will lie across a line between two strips? This problem can be solved to approximate π.

Solution: Let d be the distance from the center of the needle to the nearest line. Let θ be the acute angle that the line determined by the needle and the parallel line. Then $0 \le d \le 1/2$ and $0 \le \theta \le \pi/2$. We see that the needle lies across the nearest line if and only if

$$\frac{d}{\sin\theta} \le 1/2.$$

Let

$$E = \left\{ (d,\theta) : \frac{d}{\sin\theta} \le 1/2 \right\}.$$

Then

$$area\,(E) = \int_0^{\pi/2} \frac{1}{2}\sin\theta d\theta = 1/2$$

and

$$P\,(E) = \frac{2}{\pi}.$$