Week 3
Sept. 17 – Sept. 21

Lecture 6. Buffon's needle
Review:
Last week, the sample space is $\mathbb{R}$ or a subset of $\mathbb{R}$.
Let X be a random variable with $X \in \mathbb{R}$.
The *cumulative distribution function (cdf)* is defined as $F_X(x) = \mathbb{P}(X \leq x)$ which satisfies

$$\begin{aligned} \lim_{x\to+\infty} F_X(x) &= 1, \ \lim_{x\to-\infty} F_X(x) = 0 \\ x_1 &\leq x_2 \Longrightarrow F(x_1) \leq F(x_2) \end{aligned}$$

Let the *probability density function (pdf)* of X be f, then

$$F_X(a) = \mathbb{P}(X \leq a) = \int_{-\infty}^{a} f(x)\,dx.$$

If f is continuous at a, then

$$\frac{dF_X(x)}{dx}|_{x=a} = f(a).$$

The expectation of $g(X)$ is defined as

$$\mathbb{E}g(X) = \int g(x) f(x)\,dx.$$

Example 1: Assume that

$$f(x) = \begin{cases} 5\exp(-5x) & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Question: (i) cdf? (ii) $\mathbb{E}X^2$?

Today we go a little bit further to consider higher dimensional sample space: $\mathbb{R}^2$.
Example 2:
Choose independently two numbers X and Y uniformly from $[0, 1]$.
Question: what is the probability of $X \leq 1/2$ and $Y \leq 1/3$? Define $Z = (X, Y)$. Then equivalently we ask what is the probability of Z chosen from the region $[0, 1/2] \times [0, 1/3]$.
Question: what is the probability that $Y \leq X^2$?
More generally, for a function $f(x, y) \geq 0$ and $\int\int f(x, y)\,dxdy = 1$ we may define a probability as follows

$$\mathbb{P}(E) = \int\int_E f(x, y)\,dxdy, \ E \in \mathbb{R}^2$$

If $E=[a,b]\times[c,d]$, then

$$\mathbb{P}(E)=\int_c^d\int_a^b f(x,y)\,dxdy.$$

We say (X,Y) has density $f(x_1,x_2)$, if

$$\mathbb{P}(Z\in E)=\int\int_E f(x,y)\,dxdy.$$

The joint cumulative distribution function is defined as

$$F(a,b)=\int_{-\infty}^b\int_{-\infty}^a f(x,y)\,dxdy$$

and the expectation of $g(X,Y)$ is defined as

$$\mathbb{E}g(X,Y)=\int\int g(x,y)\,f(x,y)\,dxdy.$$

Question: Let $\Omega=[0,\pi/2]\times[0,1/2]$ and E be a set in Ω. How do you define the uniform distribution on Ω?

$$\mathbb{P}(E)=\frac{4}{\pi}\int\int_E dxdy?$$

Example 3: In mathematics, there is a long history of numerical approximations of π, an irrational number. Today we use a probabilistic approach to approximate π, using the uniform distribution introduced above.

Buffon's needle.

In mathematics, Buffon's needle problem is a question first posed in the 18th century by Georges-Louis Leclerc, Comte de Buffon: suppose we have a floor made of parallel strips of wood, each the same width 1, and we drop a needle with length $l=1$ onto the floor. What is the probability that the needle will lie across a line between two strips? This problem can be solved to approximate π.

Let d be the distance from the center of the needle to the nearest line. Let θ be the acute angle that the line determined by the needle and the parallel line. Then $0\le d\le 1/2$ and $0\le\theta\le\pi/2$. We see that the needle lies across the nearest line if and only if

$$\frac{d}{\sin\theta}\le 1/2.$$

Let

$$E=\left\{(d,\theta):\frac{d}{\sin\theta}\le 1/2\right\}.$$

Then

$$area\,(E) = \int_0^{\pi/2} \frac{1}{2} \sin\theta d\theta = 1/2$$

and

$$\mathbb{P}\,(E) = \frac{2}{\pi}.$$

Thus an approximate value of π is

$$\frac{2}{\text{the fraction of needles which lie across a line}}$$

Lecture 7. Discrete conditional distribution.

PROBLEMS OF POINTS: Two players each stake 32 pistoles on a game of three points. The players have equal chances of winning each point, and the first to win 3 points win the other's stake.

Suppose that the first player has gained two points and the second player has gained one point, but the players separate without finishing the game. How much should the first player receive from the second.

Fermat's solution to the problem of points: What is the chance for the second player to win? This is related to what is the chance for the second player to win the next point!

Example 1: Flip a coin twice, the sample space is

$$\Omega = \{HH, HT, TH, TT\}.$$

What is chance to see the outcome (or the event)$\{HH\}$?

Define

$$E = \{HH, HT\}$$

which is the event with at least one head. Suppose that we are told E has occurred. What is chance to see the outcome (or the event)$\{HH\}$?

Question: what is the probability that a family of two children has (i) two boys given that the first child is a boy? (ii) two boys given that it has at least one boy?

Define:

$$\mathbb{P}(F|E) = \frac{\mathbb{P}(F \cap E)}{\mathbb{P}(E)}$$

PROBLEMS OF POINTS is based solved by this formula? Let the conditioning event be E. Let E_i be the event that the second player wins the i^{th} point. Then

$$\mathbb{P}(E_4 \cap E_5|E) = \mathbb{P}(E_4 \cap E_5|E \cap E_4)\,\mathbb{P}(E_4|E) = 1/4.$$

Bayes formula:

Question: Suppose

$$\Omega = E_1 \cup E_2 \cup \cdots \cup E_n,\ E_i \cap E_j = \phi \text{ for } i \neq j$$

then

$$\mathbb{P}(E) = \sum_{i=1}^{n} \mathbb{P}(E|E_i)\,\mathbb{P}(E_i)\,?$$

Question:

$$\mathbb{P}(E_i|E) = \frac{\mathbb{P}(E|E_i)\,\mathbb{P}(E_i)}{\sum_{i=1}^{n} \mathbb{P}(E|E_i)\,\mathbb{P}(E_i)}?$$

Example 2:

A doctor gives a patient a test for a particular cancer. Before the results of the test, the only evidence the doctor has to go on is that 1 woman in 1000 has this cancer. Experience has shown that, in 99% of the cases in which cancer is present, the test is positive; and in 95% of the cases in which is present, it is negative. If the test turns out to be positive, what probability should the doctor assign to the event that cancer is present?

Mathematical description:

$$\begin{aligned} P\left(cancer\right) &= 1/1000, \\ P\left(+|cancer\right) &= 99\%, \text{ or } P\left(-|cancer\right) = 1\% \\ P(-|not\ cancer) &= 95\%, \text{ or } P(+|not\ cancer) = 5\% \end{aligned}$$

The question is

$$P\left(cancer|+\right) = ?$$

Example 3: The prisoner's dilemma

Three prisoners, A, B, and C, with apparently equally good records have applied for parole. The parole board has decided to release two of the three, and the prisoners know this but not which two. A warder friend of prisoner A knows who are to be released. Prisoner A realizes that it would be unethical to ask the warder if he, A, is to be released, but thinks of asking for the name of one prisoner other than himself who is to be released. He thinks that before he asks, his chances of release are 2/3. He thinks that if the warder says "B will be released," his own chances have now gone down to 1/2, because either A and B or B and C are to be released. And so A decides not to reduce his chances by asking. However, A is mistaken in his calculations. Explain.

Let X be the executionee. Let E be the event that the warder names B. We need to calculate

$$\begin{aligned} \mathbb{P}\left(X = A|E\right) &= \frac{\mathbb{P}\left(A\right)\mathbb{P}\left(E|A\right)}{\mathbb{P}\left(A\right)\mathbb{P}\left(E|A\right) + \mathbb{P}\left(B\right)\mathbb{P}\left(E|B\right) + \mathbb{P}\left(C\right)\mathbb{P}\left(E|C\right)} \\ &= \frac{1/3 \cdot 1/2}{1/3 \cdot 1/2 + 1/3 + 1/3 \cdot 0} = 1/3. \end{aligned}$$

Lecture 8. Gambler's ruin.

5.1

Two gamblers, with fortunes \$3 and \$7, bet on tosses of a fair coin. The first gambler wins \$1 if a toss gives heads; the second wins \$1 if a toss give tails. The game stops if either runs out of money. What is the probability that the first gambler wins the \$7?

Let P_F be the probability that the first gambler wins if he starts with F and his opponent starts with $10 - F$.

Note that $P_F = 1$, if $F = 10$; and $P_F = 0$, if $F = 0$.

For $1 \leq F \leq 10$, then

$$P_F = 1/2P_{F-1} + 1/2P_{F+1}.$$

So

$$P_2 = 2P_1,\ P_3 = 3P_1,\ ...,\ P_{10} = 10P_1.$$

That implies

$$P_1 = 1/10 \text{ and } P_3 = 3/10.$$

Remark: Let $Q_F = P_{F+1} - P_F$, then

$$Q_9 = Q_8 = \cdots = Q_0 = P_1$$

and

$$P_F = Q_{F-1} + \cdots + Q_0 = FP_1.$$

5.2

We assume that the coin is not fair and the probability of a head is $p \neq 1/2$. Then

$$P_F = (1-p)\,P_{F-1} + pP_{F+1}.$$

Let

$$Q_F = P_{F+1} - P_F$$

then

$$Q_0 = P_1$$

and

$$pQ_F = (1-p)\,Q_{F-1}.$$

Thus

$$Q_F = \left(\frac{1-p}{p}\right)^F P_1$$

We then have

$$P_F = \frac{\left(\frac{1-p}{p}\right)^F - 1}{\left(\frac{1-p}{p}\right)^{10} - 1}?$$

When $p = 1/3$, then $P_F = 0.008$.

5.5

You enter a casino with \$1 in your pocket. You stake \$1. Your return is \$2 with probability $p > 1/2$; you lose \$1 with probability $1 - p < 1/2$. What is the probability you play forever? (This really means the casino goes broke.)

$$P_F = 2 - 1/p?$$