Week 4
Sept. 24 – Sept. 28

Lecture 9
Random walks: Gambler's ruin.
Example 1:
Two gamblers, with fortunes \$3 and \$7, bet on tosses of a fair coin. The first gambler wins \$1 if a toss gives heads; the second wins \$1 if a toss give tails. The game stops if either runs out of money. What is the probability that the first gambler wins the \$7?

Solution 1:
Let P_F be the probability that the first gambler wins if he starts with F and his opponent starts with $10 - F$:

$$P(\text{first gambler wins the \$7}|\text{first gambler has \$}F)$$

Note that $P_F = 1$, if $F = 10$; and $P_F = 0$, if $F = 0$.
For $1 \leq F \leq 10$, then

$$P_F = 1/2P_{F-1} + 1/2P_{F+1}.$$

Let $Q_F = P_{F+1} - P_F$, then

$$Q_9 = Q_8 = \cdots = Q_0 = P_1$$

and
$$P_F = Q_{F-1} + \cdots + Q_0 = FP_1.$$

Write
$$P_2 = 2P_1,\ P_3 = 3P_1,\ ...,\ P_{10} = 10P_1.$$

That implies
$$P_1 = 1/10 \text{ and } P_3 = 3/10.$$

Solution 2: Let P_3 be the probability that the first gambler wins if he starts with \$3. Let X_∞ the be money of the first gambler finally

$$X_\infty = \begin{cases} 10, & P_3 \\ 0, & 1 - P_3 \end{cases}.$$

It is expected that $\mathbb{E}X_\infty = 3$ which implies $P_3 = 3/10$.

Example 2:
We assume that the coin is not fair and the probability of a head is $p \neq 1/2$. Then
$$P_F = (1-p)\,P_{F-1} + pP_{F+1}.$$

Let
$$Q_F = P_{F+1} - P_F$$

then

$$Q_0 = P_1$$

and

$$pQ_F = (1-p)\,Q_{F-1}.$$

Thus

$$Q_F = \left(\frac{1-p}{p}\right)^F P_1$$

We then have

$$P_F = \frac{\left(\frac{1-p}{p}\right)^F - 1}{\left(\frac{1-p}{p}\right)^{10} - 1}?$$

When $p = 2/3$, then $P_3 = .876$.

Example 2 (revisited):

You enter a casino with \$1 in your pocket. You stake \$1. Your return is \$2 with probability $p > 1/2$; you lose \$1 with probability $1 - p < 1/2$.

Assume that the total money of the casino is \$2000 billion (or trillion). If the casino lets you play long enough, what is the probability you will eventually own the casino?

$$P_1 \approx 1 - \frac{1-p}{p} = 2 - 1/p$$

which is $1/2$ when $p = 2/3$.

What is the probability you play forever if the casino has infinite money? (This really means the casino goes broke.)

$$P_F = 2 - 1/p?$$

Lecture 10. Independence

Definition.

Let $P(E) > 0$ and $P(F) > 0$. Two events E and F are called *independent* if

$$\begin{aligned} P(F|E) &= P(F) \\ P(E|F) &= P(E). \end{aligned}$$

Question: Let $P(E) > 0$ and $P(F) > 0$, then

$$E \text{ and } F \text{ are independent} \Longleftrightarrow P(E \cap F) = P(E)\, P(F)$$

Example: It is often, but not always, intuitively clear that when two events are independent. Let E be the event that"the first toss is a head" and F the event "the two outcomes are the same." Then

$$P(F|E) = P(F).$$

Two random variables X and Y are *independent* if

$$P(X \leq x, Y \leq y) = P(X \leq x) \cdot P(X \leq y)$$

for each pair x and y.

Discrete cases. The discrete random variables X_1, $X_2, \cdots$, X_n are mutually independent if

$$\begin{aligned} & P(X_1 \leq x_1, X_2 \leq x_2, \cdots, X_n \leq x_n) \\ = \; & P(X_1 \leq x_1) \cdot P(X_2 \leq x_2) \cdot \ldots \cdot P(X_n \leq x_n) \end{aligned}$$

Question: Equivalently the discrete random variables X_1, $X_2, \cdots$, X_n are mutually independent if

$$\begin{aligned} & P(X_1 = x_1, X_2 = x_2, \cdots, X_n = x_n) \\ = \; & P(X_1 = x_1) \cdot P(X_2 = x_2) \cdot \ldots \cdot P(X_n = x_n) \end{aligned}$$

It is enough to consider the case of two random variables. Let X take values $a_{1,}a_2, \ldots, a_n$, and $a \leq a_2 \leq \ldots \leq a_n$. Let Y take values $b_{1,}b_2, \ldots, b_m$, and $b_1 \leq b_2 \leq \ldots \leq b_m$. Then

$$\begin{aligned} & P(X = a_i, Y = b_j) \\ = \; & P(X \leq a_i, Y \leq b_j) - P(X \leq a_{i-1}, Y \leq b_j) - P(X \leq a_i, Y \leq b_{j-1}) \\ & + P(X \leq a_{i-1}, Y \leq b_{j-1}) \\ = \; & P(X \leq a_i)\, P(Y \leq b_j) - P(X \leq a_{i-1})\, P(Y \leq b_j) \\ & - P(X \leq a_i)\, P(Y \leq b_{j-1}) + P(X \leq a_{i-1})\, P(Y \leq b_{j-1}) \\ = \; & (P(X \leq a_i) - P(X \leq a_{i-1})) \cdot (P(Y \leq b_j) - P(X \leq Y_{j-1})) = \ldots. \end{aligned}$$

Continuous cases. The continuos random variables $X_1,\ X_2, \cdots,\ X_n$ are mutually independent if

$$\begin{aligned} & P\left(X_1 \leq x_1, X_2 \leq x_2, \cdots, X_n \leq x_n\right) \\ = \ & P\left(X_1 \leq x_1\right) \cdot P\left(X_2 \leq x_2\right) \cdot \ldots \cdot P\left(X_n \leq x_n\right) \end{aligned}$$

Question: Equivalently the continuous random variables $X_1,\ X_2, \cdots,\ X_n$ are mutually independent if and only if

$$f\left(x_1, x_2, \ldots, x_n\right) = f_{X_1}\left(x_1\right) \cdot f_{X_2}\left(x_2\right) \cdot \ldots \cdot f_{X_n}\left(x_n\right)$$

It is enough to consider the case $n = 2$. Let X and Y be independent, then

$$\int_{-\infty}^{y} \int_{-\infty}^{x} f\left(t, s\right) dt ds = \int_{-\infty}^{x} f_X\left(t\right) dt \int_{-\infty}^{y} f_Y\left(s\right) ds.$$

Example

A game of darts involves throwing a dart at circle target of unit radius. Suppose we throw a dart so that it hits the target, and we observe where it lands. Assume that the position of the dart (X, Y) is uniform on $\left\{(x, y) : x^2 + y^2 \leq 1\right\}$. You may use the polar coordinate (R, Θ) to locate your dart, i.e.,

$$X = R \sin \Theta,\ Y = R \cos \Theta.$$

Question: we see

$$area\left(R \leq r, \Theta \leq \theta\right) = \frac{r^2 \theta}{2}.$$

or

$$P\left(R \leq r, \Theta \leq \theta\right) = \frac{r^2 \theta}{2\pi}.$$

Are R and Θ independent?

Lecture 11. Conditional distribution and Conditional density.

Review: Discrete random variables X_1, $X_2, \cdots$, X_n are mutually independent if

$$\begin{aligned} & P\left(X_1 = x_1, X_2 = x_2, \cdots, X_n = x_n\right) \\ = \quad & P\left(X_1 = x_1\right) \cdot P\left(X_2 = x_2\right) \cdot \ldots \cdot P\left(X_n = x_n\right) \end{aligned}$$

Continuous random variables X_1, $X_2, \cdots$, X_n are mutually independent if and only if

$$f\left(x_1, x_2, \ldots, x_n\right) = f_{X_1}\left(x_1\right) \cdot f_{X_2}\left(x_2\right) \cdot \ldots \cdot f_{X_n}\left(x_n\right)$$

Question: Let $P\left(X = a_i, Y = b_j\right) = c_{ij}$, and

(X, Y)	$Y = b_1$	$Y = b_2$
$X = a_1$	$c_{11} = 1/12$	$c_{12} = 1/6$
$X = a_2$	$c_{21} = 1/4$	$c_{22} = 1/2$

Are X and Y independent?

Example 1: A game of darts involves throwing a dart at circle target of unit radius. Suppose we throw a dart so that it hits the target, and we observe where it lands. Assume that the position of the dart (X, Y) is uniform on $\left\{(x, y) : x^2 + y^2 \leq 1\right\}$. You may use the polar coordinate (R, Θ) to locate your dart, i.e.,

$$X = R\sin\Theta,\ Y = R\cos\Theta.$$

Are R and Θ are independent? Hint:

$$P\left(R \leq r, \Theta \leq \theta\right) = \frac{r^2\theta}{2\pi}$$

Another way to solve the problem

$$P\left(R \leq r | \Theta \leq \theta\right) = \frac{\frac{r^2\theta}{2\pi}}{\frac{\theta}{2\pi}} = r^2.$$

Conditional distribution: Discrete cases.

Let X take values $a_{1,}a_2, \ldots, a_n$. Let Y take values $b_{1,}b_2, \ldots, b_m$. Let

$$P\left(X = a_i, Y = b_j\right) = c_{ij}$$

Question:

$$P\left(X = a_i | Y = b_j\right) = \frac{c_{ij}}{\sum_k c_{kj}} = \frac{P\left(X = a_i, Y = b_j\right)}{\sum_k P\left(X = a_k, Y = b_j\right)}?$$

Conditional distribution: Continuous cases.

Example 2: Let X have density

$$f_X\left(x\right) = \lambda\exp\left(-\lambda x\right).$$

Question:

$$P(X \geq x) = \exp(-\lambda x)\,?$$

Let

$$Y = X - a$$

then

$$P(Y \geq x | X \geq x) = ?$$

Conditional density: Continuous cases.
Example 1 (continue):

$$f_{R,\Theta}((r,\theta)\,|0 \leq \theta \leq \pi/2) = ?$$

Example 1 (continue):
Question:

$$P(\pi/2 \leq \theta \leq \pi/2 + 0.01 | R = r) = ?$$

Let $f_{\Theta|R}(\theta|r)$ denote density of given $R = r$. Then

$$f_{\Theta|R}(\theta|r) = \frac{1}{2\pi} I_{[0,2\pi]}(\theta)\,.$$

Example: Two random variables X and Y with density

$$f(x,y) = \begin{cases} ye^{-xy} & x \geq 0,\ 0 < y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Question:

$$P(0 \leq X \leq 1/2 | Y = 1/2) = ?$$

or

$$P(x \leq X \leq x + \delta | Y = y) = ?$$

We should have

$$\begin{aligned} \frac{f(x,y)\,\delta\epsilon}{f_Y(y)\,\epsilon} &\approx \frac{P(x \leq X \leq x+\delta, y \leq Y \leq y+\epsilon)}{P(y \leq Y \leq y+\epsilon)} \\ &\approx P(x \leq X \leq x+\delta | Y = y) = f_{X|Y}(x|y)\,\delta \end{aligned}$$

Definition:The conditional density of X given $Y = y$ is defined as

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{f(x,y)}{\int f(x,y)\,dx}.$$

Question: If X and Y are independent, the definition of conditional density implies

$$f_{X|Y}(x|y) = f_X(x)?$$