

## Week 5

Oct. 1 – Oct. 5

### Lecture 12. Conditional distribution and Conditional density.

**Review:** Discrete random variables  $X_1, X_2, \dots, X_n$  are mutually independent if

$$\begin{aligned} & P(X_1 = x_1, X_2 = x_2, \dots, X_n = x_n) \\ &= P(X_1 = x_1) \cdot P(X_2 = x_2) \cdot \dots \cdot P(X_n = x_n) \end{aligned}$$

Continuous random variables  $X_1, X_2, \dots, X_n$  are mutually independent if and only if

$$f(x_1, x_2, \dots, x_n) = f_{X_1}(x_1) \cdot f_{X_2}(x_2) \cdot \dots \cdot f_{X_n}(x_n)$$

*Question:* Let  $P(X = a_i, Y = b_j) = c_{ij}$ , and

$$\begin{array}{lll} (X, Y) & Y = b_1 & Y = b_2 \\ X = a_1 & c_{11} = 1/12 & c_{12} = 1/6 \\ X = a_2 & c_{21} = 1/4 & c_{22} = 1/2 \end{array}$$

Are  $X$  and  $Y$  independent?

**Example 1:** A game of darts involves throwing a dart at circle target of unit radius. Suppose we throw a dart so that it hits the target, and we observe where it lands. Assume that the position of the dart  $(X, Y)$  is uniform on  $\{(x, y) : x^2 + y^2 \leq 1\}$ . You may use the polar coordinate  $(R, \Theta)$  to locate your dart, i.e.,

$$X = R \sin \Theta, Y = R \cos \Theta.$$

Are  $R$  and  $\Theta$  are independent? Hint:

$$P(R \leq r, \Theta \leq \theta) = \frac{r^2 \theta}{2\pi}$$

Another way to solve the problem

$$P(R \leq r | \Theta \leq \theta) = \frac{\frac{r^2 \theta}{2\pi}}{\frac{\theta}{2\pi}} = r^2.$$

#### Conditional distribution: Discrete cases.

Let  $X$  take values  $a_1, a_2, \dots, a_n$ . Let  $Y$  take values  $b_1, b_2, \dots, b_m$ . Let

$$P(X = a_i, Y = b_j) = c_{ij}$$

*Question:*

$$P(X = a_i | Y = b_j) = \frac{c_{ij}}{\sum_k c_{kj}} = \frac{P(X = a_i, Y = b_j)}{\sum_k P(X = a_k, Y = b_j)}?$$

#### Conditional distribution: Continuous cases.

**Example 2:** Let  $X$  have density

$$f_X(x) = \lambda \exp(-\lambda x).$$

*Question:*

$$P(X \geq x) = \exp(-\lambda x)?$$

Let

$$Y = X - a$$

then

$$P(Y \geq x | X \geq x) = ?$$

**Conditional density: Continuous cases.**

**Example 1 (continue):**

$$f_{R,\Theta}((r, \theta) | 0 \leq \theta \leq \pi/2) = ?$$

**Example 1 (continue):**

*Question:*

$$P(\pi/2 \leq \theta \leq \pi/2 + 0.01 | R = r) = ?$$

Let  $f_{\Theta|R}(\theta|r)$  denote density of given  $R = r$ . Then

$$f_{\Theta|R}(\theta|r) = \frac{1}{2\pi} I_{[0,2\pi]}(\theta).$$

**Example:** Two random variables  $X$  and  $Y$  with density

$$f(x, y) = \begin{cases} ye^{-xy} & x \geq 0, 0 < y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

*Question:*

$$P(0 \leq X \leq 1/2 | Y = 1/2) = ?$$

or

$$P(x \leq X \leq x + \delta | Y = y) = ?$$

We should have

$$\begin{aligned} \frac{f(x, y) \delta \epsilon}{f_Y(y) \epsilon} &\approx \frac{P(x \leq X \leq x + \delta, y \leq Y \leq y + \epsilon)}{P(y \leq Y \leq y + \epsilon)} \\ &\approx P(x \leq X \leq x + \delta | Y = y) = f_{X|Y}(x|y) \delta \end{aligned}$$

**Definition:** The conditional density of  $X$  given  $Y = y$  is defined as

$$f_{X|Y}(x|y) = \frac{f(x, y)}{f_Y(y)} = \frac{f(x, y)}{\int f(x, y) dx}.$$

**Example?**

*Question:* If  $X$  and  $Y$  are independent, the definition of conditional density implies

$$f_{X|Y}(x|y) = f_X(x)?$$

### Lecture 13

#### Permutations.

Let  $A$  be a set with  $n$  elements,  $A = \{a_1, a_2, a_3, \dots, a_n\}$ . For instance  $A = \{1, 2, 3, \dots, n\}$ . There are many ways to order these  $n$  elements. Each order corresponds a one to one mapping of  $A$  onto itself which is called a permutation. Then the number of permutations (or ordering) is

$$n! = n \cdot (n - 1) \cdot \dots \cdot 1.$$

#### Stirling Formula.

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$$

#### Birthday problem

**Example:** In a class of 365 students, what is the probability that at least two people in the room will have the same birthday.

Solution: Total number of sequences of birthdays is  $365^{365}$ . Total number of sequences of birthdays with no duplication is  $365 \cdot (365 - 1) \cdot \dots \cdot 2 \cdot 1$ .

#### $k$ -Permutations.

A  $k$ -permutation of  $A$  is an ordered listing of a subset of  $A$  of size  $k$ . Then the number of  $k$ -permutations (or ordering) is

$$\begin{aligned} & n \cdot (n - 1) \cdot \dots \cdot (n - (k - 1)) \\ &= \frac{n!}{(n - k)!} \end{aligned}$$

Question: In a class of 40 students, what is the probability that at least two people in the room will have the same birthday.

“**n choose j**” $-\binom{n}{k}$ .

The number of distinct subsets with  $j$  elements chosen from  $A$  is denoted by  $\binom{n}{k}$ .

Question:

$$\binom{n}{k} = \frac{n!}{(n - k)!k!}$$

**Binomial distribution.** Flip a coin  $n$  times independently. Let

$$X_i = \begin{cases} 1 & \text{ith flip is a head, } P(X_i = 1) = p \\ 0 & \text{otherwise, } P(X_i = 0) = 1 - p \end{cases}$$

Let  $X = \sum_{i=1}^n X_i$ , total number of heads. Then

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}.$$

Poker hand. Example 3.6.?.

## Lecture 14

### Random walks – Always coming back.

#### 1 dimensional case.

Let  $X_1, X_2, \dots, X_n$  be a sequence of independent random variables with  $P(X_i = 1) = P(X_i = -1) = 1/2$ .

$$P(S_{2n} = 0) = \binom{2n}{n} \left(\frac{1}{2}\right)^{2n} \sim \frac{1}{\sqrt{\frac{\pi n}{2}}}$$

Let  $P_0$  be the probability that the walk never returns to 0. Let  $E$  be the event that the walk return to 0 finitely often. Then

$$P(E) = \sum_{n=0}^{\infty} P(S_{2n} = 0) \cdot P_0.$$

*Question:*

$$\sum_{n=0}^{\infty} P(S_{2n} = 0) = +\infty?$$

#### 2 dimensional case.

$$P(E) = \sum_{n=0}^{\infty} P(S_{2n} = 0, T_{2n} = 0) \cdot P_0$$

and

$$P(S_{2n} = 0, T_{2n} = 0) \sim \left(\frac{1}{\sqrt{\frac{\pi n}{2}}}\right)^2$$

*Question:*

$$\sum_{n=0}^{\infty} P(S_{2n} = 0, T_{2n} = 0) = +\infty?$$

#### 3 dimensional case.

$$P(S_{2n} = 0, T_{2n} = 0, U_{2n} = 0) \sim \left(\frac{1}{\sqrt{\frac{\pi n}{2}}}\right)^3.$$

And

$$\sum \{S_{2n} = 0, T_{2n} = 0, U_{2n} = 0\}$$

is the number of returns to 0. If  $P_0 = 0$ , the random walk returns infinitely often.

*Question:*

$$\sum_{n=0}^{\infty} P(S_{2n} = 0, T_{2n} = 0, U_{2n} = 0) = +\infty?$$