
Review: Discrete random variables $X_1, X_2, \ldots, X_n$ are mutually independent if

$$P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n) = P(X_1 = x_1) \cdot P(X_2 = x_2) \cdots P(X_n = x_n)$$

Continuous random variables $X_1, X_2, \ldots, X_n$ are mutually independent if and only if

$$f(x_1, x_2, \ldots, x_n) = f_{X_1}(x_1) \cdot f_{X_2}(x_2) \cdots f_{X_n}(x_n)$$

Question: Let $P(X = a_i, Y = b_j) = c_{ij}$, and

$$\begin{align*}
(X, Y) & \quad Y = b_1 \\
X = a_1 & \quad c_{11} = 1/12 \\
X = a_2 & \quad c_{21} = 1/4 \\
& \quad c_{12} = 1/6 \\
& \quad c_{22} = 1/2
\end{align*}$$

Are $X$ and $Y$ independent?

Example 1: A game of darts involves throwing a dart at circle target of unit radius. Suppose we throw a dart so that it hits the target, and we observe where it lands. Assume that the position of the dart $(X, Y)$ is uniform on $\{ (x, y) : x^2 + y^2 \leq 1 \}$. You may use the polar coordinate $(R, \Theta)$ to locate your dart, i.e.,

$$X = R \sin \Theta, \quad Y = R \cos \Theta.$$ 

Are $R$ and $\Theta$ independent? Hint:

$$P(R \leq r, \Theta \leq \theta) = \frac{r^2 \theta}{2\pi}$$

Another way to solve the problem

$$P(R \leq r | \Theta \leq \theta) = \frac{\frac{r^2 \theta}{2\pi}}{\frac{\theta}{2\pi}} = r^2.$$ 

Conditional distribution: Discrete cases.

Let $X$ take values $a_1, a_2, \ldots, a_n$. Let $Y$ take values $b_1, b_2, \ldots, b_m$. Let

$$P(X = a_i, Y = b_j) = c_{ij}$$

Question:

$$P(X = a_i | Y = b_j) = \frac{c_{ij}}{\sum_k c_{kj}} = \frac{P(X = a_i, Y = b_j)}{\sum_k P(X = a_k, Y = b_j)}$$

Conditional distribution: Continuous cases.
Example 2: Let $X$ have density

$$f_X(x) = \lambda \exp(-\lambda x).$$

**Question:**

$$P(X \geq x) = \exp(-\lambda x)?$$

Let

$$Y = X - a$$

then

$$P(Y \geq x | X \geq x)?$$

**Conditional density: Continuous cases.**

Example 1 (continue):

$$f_{R,\theta}((r, \theta)|0 \leq \theta \leq \pi/2) = ?$$

Example 1 (continue):

**Question:**

$$P(\pi/2 \leq \theta \leq \pi/2 + 0.01|R = r) = ?$$

Let $f_{\theta|R}(\theta|r)$ denote density of given $R = r$. Then

$$f_{\theta|R}(\theta|r) = \frac{1}{2\pi} I_{[0,2\pi]}(\theta).$$

**Example:** Two random variables $X$ and $Y$ with density

$$f(x, y) = \begin{cases} y e^{-xy} & x \geq 0, \ 0 < y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

**Question:**

$$P(0 \leq X \leq 1/2|Y = 1/2) = ?$$

or

$$P(x \leq X \leq x + \delta|Y = y) = ?$$

We should have

$$\frac{f(x, y) \delta \epsilon}{f_Y(y) \epsilon} \approx \frac{P(x \leq X \leq x + \delta, y \leq Y \leq y + \epsilon)}{P(y \leq Y \leq y + \epsilon)} \approx P(x \leq X \leq x + \delta|Y = y) = f_{X|Y}(x|y) \delta$$

**Definition:** The conditional density of $X$ given $Y = y$ is defined as

$$f_{X|Y}(x|y) = \frac{f(x, y)}{f_Y(y)} = \frac{f(x, y)}{\int f(x, y) \, dx}.$$

**Example?**

**Question:** If $X$ and $Y$ are independent, the definition of conditional density implies

$$f_{X|Y}(x|y) = f_X(x)?$$
Lecture 13

Permutations.

Let $A$ be a set with $n$ elements, $A = \{a_1, a_2, a_3, \ldots, a_n\}$. For instance $A = \{1, 2, 3, \ldots, n\}$. There are many ways to order these $n$ elements. Each order corresponds a one to one mapping of $A$ onto itself which is called a permutation.

Then the number of permutations (or ordering) is

$$n! = n \cdot (n-1) \cdot \cdots \cdot 1.$$

**Stirling Formula.**

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$$

**Birthday problem**

**Example:** In a class of 365 students, what is the probability that at least two people in the room will have the same birthday.

**Solution:** Total number of sequences of birthdays is $365^{365}$. Total number of sequences of birthdays with no duplication is $365 \cdot (365 - 1) \cdot \cdots \cdot 2 \cdot 1$.

**$k$–Permutations.**

A $k$–permutation of $A$ is an ordered listing of a subset of $A$ of size $k$. Then the number of $k$–permutations (or ordering) is

$$n \cdot (n-1) \cdot \cdots \cdot (n-(k-1))$$

$$= \frac{n!}{(n-k)!}.$$

**Question:** In a class of 40 students, what is the probability that at least two people in the room will have the same birthday.

“$n$ choose $j$” – $\binom{n}{k}$.

The number of distinct subsets with $j$ elements chosen from $A$ is denoted by $\binom{n}{k}$.

**Question:**

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

**Binomial distribution.** Flip a coin $n$ times independently. Let

$$X_i = \begin{cases} 1 & \text{ith flip is a head, } P(X_i = 1) = p \\ 0 & \text{otherwise, } P(X_i = 0) = 1 - p \end{cases}$$

Let $X = \sum_{i=1}^{n} X_i$, total number of heads. Then

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}.$$
Lecture 14
Random walks – Always coming back.

1 dimensional case.
Let $X_1, X_2, \ldots, X_n$ be a sequence of independent random variables with $P(X_i = 1) = P(X_i = -1) = 1/2$.

$$P(S_{2n} = 0) = \left( \frac{2n}{n} \right) \left( \frac{1}{2} \right)^{2n} \sim \frac{1}{\sqrt{n^2}}$$

Let $P_0$ be the probability that the walk never returns to 0. Let $E$ be the event that the walk return to 0 finitely often. Then

$$P(E) = \sum_{n=0}^{\infty} P(S_{2n} = 0) \cdot P_0.$$

Question:
$$\sum_{n=0}^{\infty} P(S_{2n} = 0) = +\infty?$$

2 dimensional case.

$$P(E) = \sum_{n=0}^{\infty} P(S_{2n} = 0, T_{2n} = 0) \cdot P_0$$

and

$$P(S_{2n} = 0, T_{2n} = 0) \sim \left( \frac{1}{\sqrt{2n}} \right)^2$$

Question:
$$\sum_{n=0}^{\infty} P(S_{2n} = 0, T_{2n} = 0) = +\infty?$$

3 dimensional case.

$$P(S_{2n} = 0, T_{2n} = 0, U_{2n} = 0) \sim \left( \frac{1}{\sqrt{6n}} \right)^3.$$ 

And

$$\sum \{ S_{2n} = 0, T_{2n} = 0, U_{2n} = 0 \}$$

is the number of returns to 0. If $P_0 = 0$, the random walk returns infinitely often.

Question:
$$\sum_{n=0}^{\infty} P(S_{2n} = 0, T_{2n} = 0, U_{2n} = 0) = +\infty?$$