

Week 6
Oct. 8 – Oct. 12

Lecture 15

Random walks – Always coming back.

Review: Binomial distribution.

1 dimensional case.

Let X_1, X_2, \dots, X_n be a sequence of independent random variables with $P(X_i = 1) = P(X_i = -1) = 1/2$. Let

$$S_m = \sum_{i=1}^m X_i$$

then

$$P(S_{2n} = 0) = \binom{2n}{n} \left(\frac{1}{2}\right)^{2n} \sim \frac{1}{\sqrt{\pi n}}$$

Let P_0 be the probability that the walk never returns to 0. Let E be the event that the walk return to 0 finitely often,

E = walk never returns after $n = 0$
 walk returns at $n = 2$, then never returns
 Walk returns at $n = 4$, then never returns
 ...
 $P(E) = P_0 + P(S_2 = 0)P_0 + P(S_4 = 0)P_0 + \dots$

Then

$$P(E) = \left[\sum_{n=0}^{\infty} P(S_{2n} = 0) \right] \cdot P_0.$$

Question:

$$\begin{aligned} \sum_{n=0}^{\infty} P(S_{2n} = 0) &= +\infty? \\ P_0 &= ? \\ P(E) &= ? \end{aligned}$$

Answer

$$\begin{aligned} \sum_{n=0}^{\infty} P(S_{2n} = 0) &= +\infty \\ P_0 &= 0 \\ P(E) &= 0 \end{aligned}$$

For 1 dimensional random walk

$$\begin{aligned} P_0 &= P(\text{the walk never returns to 0}) = 0 \\ P(E) &= P(\text{the walk return to 0 finitely often}) = 0 \end{aligned}$$

i.e.,

$$\begin{aligned}P(\text{the walk returns to } 0 \text{ eventually}) &= 1 \\P(\text{the walk returns to } 0 \text{ infinitely often}) &= 1.\end{aligned}$$

2 dimensional case.

You have two independent sequences of independent random variables

$$\begin{aligned}P(X_i = 1) &= P(X_i = -1) = 1/2 \\P(Y_i = 1) &= P(Y_i = -1) = 1/2\end{aligned}$$

Let

$$S_m = \sum_{i=1}^m X_i, T_m = \sum_{i=1}^m Y_i.$$

Let E be the event that the walk returns to 0 finitely often and P_0 be the probability that the walk never returns to 0, then

$$P(E) = \sum_{n=0}^{\infty} P(S_{2n} = 0, T_{2n} = 0) \cdot P_0$$

where

$$\begin{aligned}&P(S_{2n} = 0, T_{2n} = 0) \cdot P_0 \\&= P(\text{walk returns at } 2n\text{-th step, then never returns})\end{aligned}$$

Stirling formula gives

$$P(S_{2n} = 0, T_{2n} = 0) \sim \left(\frac{1}{\sqrt{\pi n}}\right)^2$$

Question:

$$\sum_{n=0}^{\infty} P(S_{2n} = 0, T_{2n} = 0) = +\infty?$$

Similar to 1-dimensional case. This implies

$$P_0 = P(\text{the walk never returns to } (0, 0)) = 0$$

or

$$P(\text{the walk returns to } (0, 0) \text{ eventually}) = 1$$

3 dimensional case.

There are three independent sequences of independent random variables

$$\begin{aligned}P(X_i = 1) &= P(X_i = -1) = 1/2 \\P(Y_i = 1) &= P(Y_i = -1) = 1/2 \\P(Z_i = 1) &= P(Z_i = -1) = 1/2\end{aligned}$$

Let

$$S_m = \sum_{i=1}^m X_i, T_m = \sum_{i=1}^m Y_i, U_m = \sum_{i=1}^m Z_i$$

then

$$P(S_{2n} = 0, T_{2n} = 0, U_{2n} = 0) \sim \left(\frac{1}{\sqrt{\pi n}}\right)^3.$$

Question:

$$\sum_{n=0}^{\infty} P(S_{2n} = 0, T_{2n} = 0, U_{2n} = 0) = +\infty?$$

Note that

$$\sum \{S_{2n} = 0, T_{2n} = 0, U_{2n} = 0\}$$

is the number of returns to 0. If $P_0 = 0$, then $P(E) = 0$, i.e.,

$$P(\text{the walk returns to } (0, 0, 0) \text{ infinitely often}) = 1.$$

That means the number of returns to 0 is always infinite which contradicts with

$$P \sum \{S_{2n} = 0, T_{2n} = 0, U_{2n} = 0\} < \infty.$$

Lecture 16
Inclusion and Exclusion Principle.

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$$

This implies

$$\begin{aligned} P(A_1 \cup A_2 \cup A_3) &= P(A_1 \cup A_2) + P(A_3) - P((A_1 \cup A_2) \cap A_3) \\ &= P(A_1 \cup A_2) + P(A_3) - P((A_1 \cap A_3) \cup (A_2 \cap A_3)) \\ &= P(A_1) + P(A_2) - P(A_1 \cap A_2) + P(A_3) \\ &\quad - [P(A_1 \cap A_3) + P(A_2 \cap A_3) - P(A_1 \cap A_3 \cap A_2)] \\ &= P(A_1) + P(A_2) + P(A_3) \\ &\quad - P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3) \\ &\quad + P(A_1 \cap A_3 \cap A_2) \end{aligned}$$

More generally we have

$$\begin{aligned} P(A_1 \cup A_2 \cup \dots \cup A_n) &= \sum_{i=1}^n P(A_i) - \sum_{1 \leq i < j \leq n} P(A_i \cap A_j) \\ &\quad + \sum_{1 \leq i < j < k \leq n} P(A_i \cap A_j \cap A_k) - \dots \\ &\quad + (-1)^{n-1} P(A_1 \cap A_2 \cap \dots \cap A_n) \end{aligned}$$

Question: how many terms in each sum of the equation above?

Hat Check problem:

a hat-check girl in a restaurant, having checked n hats, gets them hopelessly scrambled and returns them at random to the n owners as they leave. What is the probability that nobody gets his own hat back?

Hint: Define

$$A_i = \{\textit{i}th \textit{owner gets his own hat}\}$$

Then

$$\begin{aligned} P(A_i) &= \frac{1}{n} \\ P(A_i \cap A_j) &= \frac{1}{n(n-1)}, \quad i < j \\ P(A_i \cap A_j \cap A_k) &= \frac{1}{n(n-1)(n-2)}, \quad i < j < k \\ &\dots \end{aligned}$$

and

$$\begin{aligned} & P(A_1 \cup A_2 \cup \dots \cup A_n) \\ &= 1 - \frac{1}{2!} + \frac{1}{3!} - \dots + (-1)^{n-1} \frac{1}{n!} \end{aligned}$$

Question:

$$\lim_{n \rightarrow \infty} \left(1 - \frac{1}{2!} + \frac{1}{3!} - \dots + (-1)^{n-1} \frac{1}{n!} \right) = ?$$

Question:

$$\begin{aligned} \{A_1 \cup A_2 \cup \dots \cup A_n\} &= 1 - \{\bar{A}_1 \cap \bar{A}_2 \cap \dots \cap \bar{A}_n\} \\ &= 1 - \{\bar{A}_1\} \{\bar{A}_2\} \dots \{\bar{A}_n\} \\ &= 1 - (1 - \{A_1\})(1 - \{A_2\}) \dots (1 - \{A_n\}) \\ &= \sum_{i=1}^n \{A_i\} - \sum_{1 \leq i < j \leq n} \{A_i \cap A_j\} \\ &\quad + \sum_{1 \leq i < j < k \leq n} \{A_i \cap A_j \cap A_k\} - \dots \\ &\quad + (-1)^{n-1} \{A_1 \cap A_2 \cap \dots \cap A_n\} \end{aligned}$$

Lecture 17. Bernoulli and Binomial, Expectation

Review

Probability. Conditional probability. Combinatorics.

Sample space. Event. Probability. Random variable. Expectation. Conditional distribution (density). $\binom{n}{k}$.

Clarification: Let A be an event. Define

$$\{A\} = \begin{cases} 1 & \omega \in A \\ 0 & \text{otherwise} \end{cases}$$

This is a function of outcome, so it is a random variable. An alternative notation for $\{A\}$ is $1_A(\omega)$. Note that

$$\begin{aligned} P(\{A\} = 1) &= P(\omega \in A) = P(A) \\ P(\{A\} = 0) &= P(\omega \notin A) = 1 - P(A) \end{aligned}$$

then the expectation of $\{A\}$ is

$$E\{A\} = 1 \cdot P(A) + 0 \cdot (1 - P(A)) = P(A).$$

What to do this week: we describe the discrete probability distributions and continuous probability distributions that occur most often in the analysis of experiments.

Discrete uniform distribution.

We have seen some examples that all outcomes of an experiment are equally likely. Let X be a random variable representing the outcome of an experiment of this kind. Let x_1, x_2, \dots, x_m denote all possible outcomes. Then

$$\Omega = \{x_1, x_2, \dots, x_m\}$$

and

$$P(X = x_i) = \frac{1}{m}.$$

A simple but important example

$$\Omega = \{0, 1\}$$

and

$$P(X = 0) = P(X = 1) = 1/2.$$

Binomial.

It is the distribution of random variable which counts the number of heads when a coin is tossed n times. Let $X_i = 1$ if a head occurs at the i th toss, otherwise $X_i = 0$. Assume that

$$P(X_i = 1) = p, \quad P(X_i = 0) = 1 - p,$$

then the distribution of $X = X_1 + X_2 + \dots + X_n$ is

$$P(X = x) = \binom{n}{x} p^x q^{n-x}, x = 0, 1, \dots, n.$$

Question:

$$EX = ?$$

Solution 1:

$$\begin{aligned} EX &= \sum_{x=0}^n x \cdot \binom{n}{x} p^x q^{n-x} \\ &= \sum_{x=0}^n x \cdot \frac{n!}{x!(n-x)!} p^x q^{n-x} \\ &= \sum_{x=1}^n np \cdot \frac{(n-1)!}{(x-1)!(n-x)!} p^{x-1} q^{n-x} \\ &= np \sum_{x=1}^n \frac{(n-1)!}{(x-1)!(n-x)!} p^{x-1} q^{n-x} = ? \end{aligned}$$

Solution 2:

$$EX = E(X_1 + X_2 + \dots + X_n) \stackrel{?}{=} EX_1 + EX_2 + \dots + EX_n = ?$$

For solution 2 we need to answer a question: for two discrete (continuous) random variables X and Y , is it true that

$$E(X + Y) = EX + EY?$$

Proof: Let the sample space of X and Y be denoted by Ω_X and Ω_Y , and

$$\begin{aligned} \Omega_X &= \{x_1, x_2, \dots, x_i, \dots\} \\ \Omega_Y &= \{y_1, y_2, \dots, y_j, \dots\}. \end{aligned}$$

The sample space of (X, Y) is

$$\Omega_{(X,Y)} = \{(x_i, y_j), i = 1, 2, \dots, j = 1, 2, \dots\}.$$

Then

$$\begin{aligned} E(X + Y) &= \sum_i \sum_j (x_i + y_j) P(X = x_i, Y_j = y_j) \\ &= \sum_i \sum_j x_i P(X = x_i, Y_j = y_j) + \sum_i \sum_j y_j P(X = x_i, Y_j = y_j) \\ &= \sum_i x_i \sum_j P(X = x_i, Y_j = y_j) + \sum_j y_j \sum_i P(X = x_i, Y_j = y_j) \\ &= \sum_i x_i P(X = x_i) + \sum_j y_j P(Y_j = y_j) \\ &= EX + EY. \end{aligned}$$

Question:

$$E(X_1 + X_2 + \dots + X_n) = EX_1 + EX_2 + \dots + EX_n$$

Question:

$$E(cX) = cEX?$$

Question:

$$E(X - EX)^2 = EX^2 - (EX)^2?$$

Question: for two continuous random variables X and Y , is it true that

$$E(X + Y) = EX + EY?$$

The answer is exactly the same:

$$E(X + Y) = \int \int (x + y) f_{X,Y}(x, y) dx dy$$

where $f_{X,Y}(x, y)$ is the joint density of (X, Y) . Then

$$\begin{aligned} E(X + Y) &= \int \int x f_{X,Y}(x, y) dx dy + \int \int y f_{X,Y}(x, y) dx dy \\ &= \int x \left[\int f_{X,Y}(x, y) dy \right] dx + \int y \left[\int f_{X,Y}(x, y) dx \right] dy \\ &= \int x f_X(x) dx + \int y f_Y(y) dy \\ &= EX + EY. \end{aligned}$$