STAT 242/542b, Theory of Statistics, HW #1

(Ch2:9)
Let $X \sim \text{Exp}(\beta)$. Find $F(x)$ and $F^{-1}(q)$.

(Ch3:19)
This question is to help you understand the idea of a **sampling distribution.** Let $X_1, X_2, \cdots, X_n$ be i.i.d. with mean μ and variance σ^2. Let $\overline{X}_n = n^{-1}\sum_{i=1}^n X_i$. Then $\overline{X}_n$ is a **statistics**, that is, a function of the data. Since $\overline{X}_n$ is a random variable, it has a distribution. This distribution is called the sampling distribution of the statistics. Recall from Theorem 3.17 that $\mathbb{E}(\overline{X}_n) = \mu$ and $\mathbb{V}(\overline{X}_n) = \sigma^2/n$. Do not confuse the distribution of the data f_X and the distribution of the statistics $f_{\overline{X}_n}$. To make this clear, let $X_1, \cdots, X_n \sim \text{Uniform}(0,1)$. Let f_X be the density of the Uniform$(0,1)$. Plot f_X. Now let $\overline{X}_n = n^{-1}\sum_{i=1}^n X_i$. Find $\mathbb{E}(\overline{X}_n)$ and $\mathbb{V}(\overline{X}_n)$. Plot them as a function of n. Interpret. Now simulate the distribution of $\overline{X}_n$ for $n = 1, 5, 25, 100$. Check that the simulated values of $\mathbb{E}(\overline{X}_n)$ and $\mathbb{V}(\overline{X}_n)$ agree with your theoretical calculations. What do you notice about the sampling distribution of $\overline{X}_n$ as n increases?

(Ch4:1)
Let $X \sim \text{Exponential}(\beta)$. Find $\mathbb{P}(|X - \mu_X| \geq k\sigma_X)$ for $k \geq 1$. Compare this to the bound you get from Chebyshev's inequality.

(Ch5:1)
Let $X_1, \cdots, X_n$ be i.i.d. with finite mean $\mu = \mathbb{E}(X_1)$ and finite variance $\sigma^2 = \mathbb{V}(X_1)$. Let $\overline{X}_n$ be the sample mean and let S_n^2 be the sample variance.
(a) Show that $\mathbb{E}(S_n^2) = \sigma^2$.
(b) Show that $S_n^2 \to^P \sigma^2$. Hint: Show that $S_n^2 = c_n n^{-1}\sum_{i=1}^n X_i - d_n \overline{X}_n^2$ where $c_n \to 1$ and $d_n \to 1$. Apply the law of large numbers to $n^{-1}\sum_{i=1}^n X_i^2$ and to $\overline{X}_n$. Then use part (e) of Theorem 5.5.

(Ch5: 6)
Suppose the height of men has mean 68 inches and standard deviation 2.6 inches. We draw 100 men at random. Find (approximately) the probability that the average height of men in our sample will be at least 68 inches.