Lecture notes, Posterior interval.

Recall the following theorem we sketched the proof in class (see also the textbook, page 181).

**Theorem:**

\[
\theta | X \approx N \left( \hat{\theta}_{MLE}, \frac{1}{n I_1(\hat{\theta}_{MLE})} \right)
\]

**Proof** (see also page 190 of the textbook):

\[
\log f(\theta | X) \approx l(\theta) = l(\hat{\theta}_{MLE}) + \frac{l''(\hat{\theta}_{MLE})}{2} (\hat{\theta}_{MLE} - \theta)^2
\]

where \(l'(\hat{\theta}_{MLE}) = 0\) and \(-l''(\hat{\theta}_{MLE}) \approx n I_1(\hat{\theta}_{MLE})\). Then

\[
f(\theta | X) \propto \exp \left( -\frac{(\theta - \hat{\theta}_{MLE})^2}{n I_1(\hat{\theta}_{MLE})} \right)
\]

This theorem implies the posterior mean \(\hat{\theta}_B = E(\theta | X) \approx \hat{\theta}_{MLE}\).

Remark: it doesn’t make any sense saying what is the distribution of \(\hat{\theta}_B\) given \(X\), although I was mistakenly writing \(\hat{\theta}_B | X\) instead of \(\theta | X\) on the blackboard today. Given \(X\), \(\hat{\theta}_B\) is just a deterministic number.

An exact 95% posterior interval estimation: find \(a\) and \(b\) such that

\[
\int_a^b f(\theta | X) d\theta = 95\%
\]

An approximate 95% posterior interval:

\[
\hat{\theta}_B \pm 2 \frac{1}{\sqrt{n I_1(\hat{\theta}_{MLE})}}
\]

An approximate 95% confidence interval:

\[
\hat{\theta}_{MLE} \pm 2 \frac{1}{\sqrt{n I_1(\hat{\theta}_{MLE})}}
\]

What about estimating \(g(\theta)\)?

By the delta method,

\[
g(\theta) | X \approx N \left( g(\hat{\theta}_{MLE}), \frac{|g'(\hat{\theta}_{MLE})|^2}{n I_1(\hat{\theta}_{MLE})} \right)
\]
which implies \( \hat{g}_B = E(g(\theta) | X) \approx g(\hat{\theta}_{MLE}) \), and an approximate 95% posterior interval is

\[
\hat{g}_B \pm 2 \frac{|g(\hat{\theta}_{MLE})|}{\sqrt{nI_1(\hat{\theta}_{MLE})}}.
\]