

Cramer-Rao (or Information) Inequality:

Let X have pdf (or pmf) $f(x|\theta)$ with fisher information $I_X(\theta)$. Let $\delta(X)$ be an unbiased estimator of $g(\theta)$. Then

$$\text{var}(\delta(X)) \geq \frac{(g'(\theta))^2}{I_X(\theta)},$$

where the equality holds iff

$$\delta(X) = g(\theta) + \frac{g'(\theta)}{I_X(\theta)} \frac{\partial}{\partial \theta} \log f(x|\theta).$$

Example 1: Let $X = (X_1, X_2, \dots, X_n)$ with X_1, X_2, \dots, X_n i.i.d. $\sim N(\theta, 1)$. Then

$$I_X(\theta) = nI_1(\theta) = n$$

and the information inequality gives

$$\text{var}(\delta(X)) \geq \frac{1}{n}$$

for any unbiased estimator $\delta(X)$ of θ . The score function is

$$\begin{aligned} \frac{\partial}{\partial \theta} \log f(x|\theta) &= \frac{\partial}{\partial \theta} \log \left(\prod_{i=1}^n \frac{1}{\sqrt{2\pi}} \exp \left(-\frac{(x_i - \theta)^2}{2} \right) \right) \\ &= \frac{\partial}{\partial \theta} \left(\sum_{i=1}^n \left(\log \frac{1}{\sqrt{2\pi}} - \frac{(x_i - \theta)^2}{2} \right) \right) \\ &= \sum_{i=1}^n x_i - \theta. \end{aligned}$$

Thus the lower bound of the information inequality is attained iff

$$\delta(X) = \theta + \frac{1}{n} \left(\sum_{i=1}^n x_i - \theta \right) = \bar{X}.$$

Example 2: Let $X = (X_1, X_2, \dots, X_n)$ with X_1, X_2, \dots, X_n i.i.d. $\sim \text{Bernoulli}(p)$. Then

$$I_X(p) = nI_1(p) = \frac{n}{p(1-p)}$$

and the information inequality gives

$$\text{var}(\delta(X)) \geq \frac{p(1-p)}{n}$$

for any unbiased estimator $\delta(X)$ of p . The score function is

$$\frac{\partial}{\partial p} \log f(x|p) = n \left(\frac{\bar{X}}{p} - \frac{1 - \bar{X}}{1 - p} \right) = \frac{n\bar{X}}{p(1-p)} - \frac{n}{1-p}.$$

Thus the lower bound of the information inequality is attained iff

$$\delta(X) = p + \frac{p(1-p)}{n} \left(\frac{n\bar{X}}{p(1-p)} - \frac{n}{1-p} \right) = \bar{X}.$$

And the information inequality gives

$$\text{var}(\delta(X)) \geq (2p)^2 \frac{p(1-p)}{n} = \frac{4p^3(1-p)}{n}$$

for any unbiased estimator $\delta(X)$ of p^2 . Thus the lower bound of the information inequality is attained iff

$$\delta(X) = p^2 + \frac{2p^2(1-p)}{n} \left(\frac{n\bar{X}}{p(1-p)} - \frac{n}{1-p} \right) = 2p\bar{X} - p^2.$$

Note that $\delta(X)$ now depends on p . It is not an estimator. So the lower bound can not be attained.