Stat 610

Final Examination

Due Dec. 9, 2005, 2:30PM

1 (20 points). Let $X_1, X_2, \ldots, X_n$ be i.i.d. according to $Cauchy\,(\theta, 1)$. Assume that Pitman estimator δ_P is an asymptotically efficient estimator of θ with $\lim_{n\to\infty} var\left(\sqrt{n}\delta_P\right) = 2$. Find an unbiased estimator of θ^3 such that it is asymptotically efficient at $\theta \neq 0$.

Hint: you may assume the third moment of δ_P is finite.

2 (20 points). Let

$$X_i|\theta \overset{i.i.d.}{\sim} U\left(0,\theta\right),\ i = 1, 2, \ldots, n$$

and

$$\frac{1}{\theta}|a, b \sim Gamma\left(a, b\right),\ a, b \text{ known}$$

Show that

(i) The Bayes estimator will only depend on the data through $Y = \max_i X_i$.

(ii)

$$E\left(\theta|a, b\right) = \frac{1}{b\left(n + a - 1\right)} \frac{P\left(\chi^2_{2(n+a-1)} < 2/\left(by\right)\right)}{P\left(\chi^2_{2(n+a)} < 2/\left(by\right)\right)}$$

3 (20 points). Let $X \sim Bernoulli\,(p)$ with unknown $1 > p > 0$.

(i) Is X a minimax estimator of p under the squared error loss $L\left(p, \delta\right) = \left(p - \delta\right)^2$. Why?

(ii) Is X an admissible estimator of p under the squared error loss $L\left(p, \delta\right) = \left(p - \delta\right)^2$. Why?

4 (20 points). Observe $X = \left(X_1, X_2, \ldots, X_n\right)$ be $N\left(\theta, I_n\right)$ with unknown $\theta = \left(\theta_1, \theta_2, \ldots, \theta_n\right) \in R^n$. For given $p \geq 1$ and $\lambda > 0$, define

$$\delta_i\left(X\right) = \left(1 - \frac{\lambda^p}{|X_i|^p}\right)_+ X_i\ .$$

Show that

(i)

$$E_{X|\theta} \sum_{i=1}^{n} \left(\delta_i\left(X\right) - \theta_i\right)^2 = E_{X|\theta} SURE\left(\lambda\right)$$

with

$$SURE\left(\lambda\right) = n + \sum_{i=1}^{n} \left[\left(X_i^2 - 2\right) I\left(|X_i| \leq \lambda\right) + \left(\frac{2\left(p-1\right)\lambda^p}{|X_i|^p} + \frac{\lambda^{2p}}{|X_i|^{2p-2}}\right) I\left(|X_i| > \lambda\right)\right]$$

(ii)The minimum of $SURE(\lambda)$ is achieved at some $\lambda \in A$ with

$$A = \{|X_i|\,;\ 1 \leq i \leq n\} \cup \{0\}\,.$$

5 (20 points). Let $X_1,\, X_2,\, \ldots,\, X_n$ be i.i.d. with density

$$f_{\theta,\sigma}(x) = \frac{\theta}{\sigma} x^{\theta-1} e^{-x^{\theta}/\sigma},\ x > 0, \theta > 0, \sigma > 0.$$

where θ and σ are unknown. Show that

(i) The likelihood equation has a unique solution unless all X_i are equal.

(ii) The sequence of maximum likelihood estimators $\left(\widehat{\theta}_{MLE}, \widehat{\sigma}_{MLE}\right)$ is asymptotically normal.