ANALYSIS OF THE GREEDY LIST COLORING ALGORITHM ON A RANDOM GRAPH

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As a special project for Spring 2007, a graph coloring algorithm proposed by Achlioptas and Molloy (1997) is analyzed.

1. INTRODUCTION

1.1. **The Erdös-Rényi Graph.** This paper is concerned with the behavior of a graph coloring algorithm on a random graph. A random graph is a graph generated by some random process. Specifically, the random graphs that will be utilized in this paper are *Erdös-Rényi Graphs* (ER Graphs). ER Graphs are generated by creating edges between each pair of vertices with equal probability independent of other edges. In this paper, the ER Graph variant being utilized is denoted by $G(m, p)$. For this graph, m is the number of vertices, and an edge between any two vertices with probability p. On average mp edges will appear in graph, and the degree of each vertex is distributed binomial. That is, for a vertex v in the set of vertices V of an ER Graph $G(m, p)$

$$\mathbb{P}\left\{\deg(v) = k\right\} = \binom{M}{k} p^k (1-p)^{M-k} \text{ where } M = \binom{m}{2}.$$

It should be noted that, in this equation, M is is the total number of possible edges.

1.2. **The Threshold Phenomenon and Graph Coloring.** In their paper on random graphs, Erdös and Rényi (1960) prove the existence of a threshold phenomena. For a graph property Q, which exhibits the threshold phenomena, the probability $G(m, p)$ has Q increases sharply at a certain critical value of p. One of the properties which shows this behavior is the chromatic number of a graph.

Graph Coloring, or more specifically *Vertex Coloring*, refers to an assignment of colors to the vertices of a graph such that no neighbors share the same coloring. Two vertices in a graph are neighbors if they share an edge. A coloring that uses at most k colors is called a (proper) k-*coloring*.

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The minimum number of colors needed to properly color a graph is called its chromatic number and is denoted $\chi(\cdot)$.

1.3. **The Achlioptas Molloy Paper.**

In Achlioptas and Molloy (1997) a greedy graph coloring algorithm for graphs is introduced. For a graph $G = (V, E)$ with V vertices and E edges, denote L_v as the list of available colors for vertex v in V. The algorithm is given as follows:

- Initialize: $U = V$, and $L_v = \{1, ..., k\}$ for every $v \in V$.
- While $U \neq \emptyset$:
  (1) At time t select a vertex v_t uniformly at random from $\{v_t \in U : |L_v| \text{ is smallest }\}$
  (2) if $|L_v| > 0$
    - then assign to v_t a color ω chosen uniformly at random from $|L_v|$, and set $U = U \setminus \{v_t\}$
    - else exit (fail).

  (3) For each $u \in U$ that is a neighbor of v_t set $L_u = L_u \setminus \{\omega\}$

In addition to proposing this algorithm, the paper also provides an analysis its effectiveness. It should be noted that in the analysis (as with this paper) it is assumed that the random graph is not given entirely at the beginning of the algorithm. Instead, we learn about the graph as the algorithm proceeds. This constraint, proposed by (D.E. Knuth 1990), is called the *method of deferred decisions*. In the analysis of the algorithm, this method is employed so that at each step, the vertices that are exposed are those adjacent to colored vertices.

2. Model Description

In this paper we start by considering the algorithm with $k = 3$. Let the colors being considered be denoted A, B, C. For each step in the algorithm a given node is colored and the number of possible colorings for adjacent nodes is decreased. Let N denote the number of nodes whose colorings may be $A, B,$ or C. Then,

$$N = \sum_{v \in V} \{L_v = \{A, B, C\}\} .$$

Also, let X_{AB}, X_{AC}, X_{BC} denote the number of nodes whose possible list colorings are AB, AC, and BC respectively. Finally, let

$$S_t = X_{AB}(t) + X_{AC}(t) + X_{BC}.$$

That is, S_t denotes the total number of vertices with 2 possible colorings at time t.

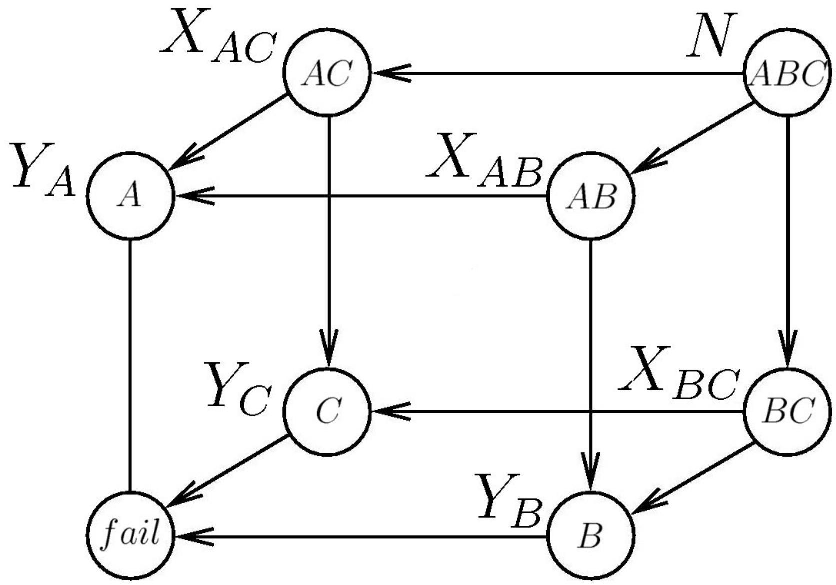

FIGURE 1. A Visualization for the List Counts

For a given graph the algorithm proceeds as follows. At the first stage, select a v_1 in V uniformly at random. Select its color uniformly at random over the three colors. Say it is C. For each of its neighbors (of which there are $b_1 \sim \text{Bin}(n - S_0, c/n)$) remove the C from its list of possible colorings and discard v_1. This causes $X_{AB}(1)$ to have b_1 items, and this causes N_1 to have $n - S_1 - 1$ items. Next, select one of the $X_{AB}(1)$ vertices uniformly at random and call it v_2. The color for v_2 is selected uniformly at random from A or B; say it is B. All of v_2's neighboring nodes have B removed from their possible colorings. This causes $X_{AC}(t)$ to receive b_2 items and N_2 to have $n - S2 - 2$ items. Depending on the connectivity of the graph, it is also possible at this stage that a small number of items could go from either X_{AB} or X_{AC} to Y_A, Y_B, or Y_C. If this is the case, then the vertex selected for coloring at the next step is chosen from one of the single-color vertices. If it is the case, that at some t in the algorithm, a number of items go from a stack of single colorings to the $fail$ state, then the algorithm fails to correctly color the graph. The algorithm continues until either all vertices in the graph are colored or it fails. A visualization of the structure of this algorithm is shown in Figure (1).

Consider the following simplification of the model. Suppose that the model in Figure (1) is restricted to have only vertices ABC, AB, AC, and BC. Also, instead of discarding a colored vertex, which reduces the total vertex count by 1, this item will remain in its respective stack.

The number of items flowing from ABC to a two-color vertex stack at time $t+1$ given the number of items in each of the two-color stacks will be modeled as

$$b_{t+1} \sim \text{Bin}(n - S_t, c/n)$$

where

$$S_t = X_{AB}(t) + X_{AC}(t) + X_{BC}(t)$$

At a given time t one of the stacks X_{AB}, X_{AC}, or X_{BC} is picked to be colored. The stack is chosen proportional to the number of elements it has. In other words,

$$\mathbb{P}\{X_{BC} \text{ will be picked at time } t+1 | X_{AB}(t), X_{AC}(t), X_{BC}(t)\} = \frac{X_{BC}(t)}{S_t}.$$

After picking one of the stacks, one of the unpicked stacks is selected to receive a number of items from X_{ABC}. Let ε_t be the indicator function telling which of the stacks will receive b_t. In other words

$$\varepsilon_t = \begin{bmatrix} \{AB \text{ receives } b_t\} \\ \{AC \text{ receives } b_t\} \\ \{BC \text{ receives } b_t\} \end{bmatrix}$$

If at time t, AB is selected for coloring, then with equal probability it will be given color A or B. This means that with equal probability, b_t will go from ABC to BC or AB respectively. Letting $\mathbb{P}_t$ denote the probability conditioned on information through time t, it can be seen that

$$\xi_t = \mathbb{P}_t \varepsilon_{t+1} = \frac{1}{2} \begin{bmatrix} \frac{X_{AC}(t)+X_{BC}(t)}{S_t} \\ \frac{X_{AB}(t)+X_{BC}(t)}{S_t} \\ \frac{X_{AC}(t)+X_{AB}(t)}{S_t} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 - \frac{X_{AB}(t)}{S_t} \\ 1 - \frac{X_{AC}(t)}{S_t} \\ 1 - \frac{X_{BC}(t)}{S_t} \end{bmatrix}.$$

If, for example, X_AB is much larger than both X_AC and X_BC, then AB is more likely to be selected for coloring. Therefore, X_{AB} is unlikely to receive more items until its size is closer to X_AC or X_BC. A more formal justification of this balancing mechanism is provided in Section 3.2.

3. Characteristics of the Simplified Model

In this section the distribution for S_t is established. Next, it is shown that counts for the two color stacks ($X_{AB}(t)$,$X_{AC}(t)$, and $X_{BC}(t)$) tend to remain close to each other over time. Finally, a bound is given on the second moment of the stack differences.

3.1. **The Distribution of S_t.** For the model described above, the distribution of the total stack counts

$$S_t = \text{Bin}(n, 1-(1-c/n)^t).$$

This can be seen by considering a single urn containing m balls. At each time t, W balls are taken from the urn where

$$W_1 \sim \text{Bin}(m, c/m)$$

or, more generally,

$$W_{t+1}|W_t \sim \text{Bin}(m - W_t, c/m).$$

We would like to get the distribution function for

$$S_t = \sum_{i=1}^{t} W_t$$

At each time there is a c/m chance that a ball in the urn will be picked to be removed. The probability that a ball is not chosen at time 1 is $1 - c/m$. The probability that a ball is not chosen at 2 is the probability of the intersection of the events that it is not chosen at time 1 and time 2, i.e. $(1 - c/m)^2$. Inductively, the probability that a ball is not chosen at time t is $(1 - c/m)^t$. Then,

$$\sum_{i=1}^{t} W_t = \text{Bin}(m, 1-(1-c/m)^t).$$

This establishes the distribution of S_t since W_t in the described urn problem is the same as S_t in the model.

3.2. X_{AB}**,** X_{AC}**, and** X_{BC} **Remain Balanced.** Let b_t be the binomial number coming down from X_{ABC} at time t. Define

$$\Delta_{t+1}X = b_{t+1}\varepsilon_{t+1}.$$

It should be noted that

$$\mathbb{P}_t\Delta_{t+1}X = \mathbb{P}_t b_{t+1}\varepsilon_{t+1} = \mathbb{P}_t b_{t+1}\mathbb{P}_t\varepsilon_{t+1} \text{ since } b_t \tag{3.1}$$

since b_t is conditionally independent of ε_t for all t.

Now, define

$$Z_t = X_{AB}(t) - X_{BC}(t).$$

Letting

$$\delta = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\begin{aligned}\mathbb{P}Z_t\Delta_{t+1}Z &= \mathbb{P}Z_t\mathbb{P}_t\Delta_{t+1}Z \\ &= \mathbb{P}Z_t\mathbb{P}_t\delta^T\varepsilon_{t+1}b_{t+1} \\ &= \mathbb{P}Z_t\mathbb{P}_t\delta^T\varepsilon_{t+1}\mathbb{P}_t b_{t+1} \text{ by Equation (3.1) .} \end{aligned} \tag{3.2}$$

Examining only $\mathbb{P}_t\delta^T\varepsilon_{t+1}$

$$\begin{aligned}\mathbb{P}_t\delta^T\varepsilon_{t+1} &= d^T\xi_t \\ &= \frac{1}{2}\left(1 - \frac{X_{AB}(t)}{S_t} - 1 + \frac{X_{AB}(t)}{S_t}\right) \\ &= \frac{1}{2}\left(-\frac{X_{AB}(t)}{S_t} + \frac{X_{AB}(t)}{S_t}\right) \\ &= -\frac{1}{2}\frac{Z_t}{S_t}. \end{aligned} \tag{3.3}$$

Substituting the results of Equation (3.3) into Equation (3.2), and recalling that b_t is conditionally distributed as binomial we get

$$\mathbb{P}Z_t\Delta_{t+1}Z = -\frac{1}{2}\mathbb{P}\frac{Z_t^2}{S_t}\frac{c}{n}(n - S_t).$$

Since $Z_t^2, S_t \geq 0$ for all t and $c, n \geq 0$,

$$\mathbb{P}Z_t\Delta_{t+1}Z \leq 0 \text{ for all } t. \tag{3.4}$$

Equation (3.4) implies that if, at some time, the difference $|X_{AB}(t) - X_{BC}(t)|$ gets large, the stack selection process ensures that it will not remain large for long since subsequent iterations of the algorithm will tend to reduce this difference. Also, since the choice of stacks used to define Z_t was arbitrary we can assume this negative feedback mechanism exists between all two-color stacks. Therefore, assuming $\mathbb{P}_tZ_{t+1}^2$ is bounded, we can conclude that the stack difference between any two stacks tends to be around zero.

3.3. $\mathbb{P}_tZ_{t+1}^2$ **is Bounded for finite** t**.** For the given model we would like to show $\mathbb{P}_tZ_{t+1}^2$ is bounded for finite t.

$$\begin{aligned}\mathbb{P}_tZ_{t+1}^2 &= \mathbb{P}_t(Z_t + \Delta_{t+1}Z)^2 \\ &= \mathbb{P}_t(Z_t^2 + Z_t\Delta_{t+1}Z + \Delta_{t+1}Z^2) \\ &= Z_t^2 + Z_t\mathbb{P}_t\Delta_{t+1}Z + \mathbb{P}_t\Delta_{t+1}Z^2 \end{aligned}$$

Therefore, if we can show that $Z_t^2 + Z_t\mathbb{P}_t\Delta_{t+1}Z$ and $\mathbb{P}_t(\Delta_{t+1}Z)^2$ are bounded $\mathbb{P}_tZ_{t+1}^2$ must also be bounded.

First we show $Z_t^2 + Z_t\Delta_{t+1}Z$ is Bounded.

$$\begin{aligned} Z_t^2 + Z_t\mathbb{P}\Delta_{t+1}Z &= Z_t^2 + Z_t\left(-\frac{1}{2}\frac{Z_t}{S_t}\frac{c}{n}(n-S_t)\right) \\ &= Z_t^2\left(1-\frac{c}{2}\left(\frac{1}{S_t}-\frac{1}{n}\right)\right) \\ &\leq Z_t^2 \text{ since } S_t \leq n \qquad (3.5) \end{aligned}$$

This should come as no surprise since it has already been established in Equation 3.4 that there is a negative feedback mechanism between Z_t and $\Delta_{t+1}Z$.
Now we show $\mathbb{P}_t(\Delta_{t+1}Z)^2$ is Bounded. To show $\mathbb{P}_t(\Delta_{t+1}Z)^2$ is bounded we first note

$$\text{VAR}_t\Delta_{t+1}Z = \text{VAR}_t d^T\Delta_{t+1}X = d^T(\text{VAR}_t\Delta_{t+1}X)d.$$

Next, for any vector λ of appropriate length,

$$\begin{aligned} \text{VAR}_t\lambda^T\Delta_{t+1}X &\leq \mathbb{P}_t\lambda^T\Delta_{t+1}X(\Delta_{t+1}X)^T\lambda \\ &= \lambda^T\mathbb{P}_t(\varepsilon_{t+1}b_{t+1})^2\lambda \\ &= \lambda^T\mathbb{P}_t\varepsilon_{t+1}^2\mathbb{P}_t b_{t+1}^2\lambda \text{ by Equation (3.1) .} \end{aligned}$$

Since b_t is distributed as binomial, we can bound its second moment as

$$\begin{aligned} \mathbb{P}_t b_{t+1}^2 &= \left(1-\frac{c}{n}\right)\frac{c}{n}(n-S_t) + \left(\frac{c}{n}\right)^2(n-S_t)^2 \\ &\leq \left(1-\frac{c}{n}\right)\frac{c}{n}n + \left(\frac{c}{n}\right)^2 n^2 \\ &\leq c + c^2. \qquad (3.6) \end{aligned}$$

Using the results of the two Inequalities (3.5) and (3.6),

$$\begin{aligned} \mathbb{P}_t Z_{t+1}^2 &\leq Z_t^2 + c + c^2 \\ &\leq Z_0^2 + t(c+c^2). \end{aligned}$$

This shows that, for finite t, the conditional second moment of Z_t will grow, at most, linearly in t. Therefore, for any finite t, the conditional second moment is bounded. This same technique can be used to bound the unconditional second moment. I.e.

$$\mathbb{P}Z_{t+1}^2 \leq Z_0^2 + t(c+c^2).$$

4. MARTINGALE BOUNDS

In this section, a martingale is constructed from the simplified model to investigate its tail bounds. The first subsection describes the construction. The following three subsections provide bounds based on the Hoeffding Inequality, the Bennett Inequality, and a Poisson approximation.

4.1. **(Martingale and Inequality Construction).** For the described simplified model

$$M_{t+1} = M_t + \Delta_{t+1} Z - \mathbb{P}_t \Delta_{t+1} Z$$

M_t is a martingale.

Proof.

$$\begin{aligned}
\mathbb{P}_t M_{t+1} &= \mathbb{P}_t (M_t + \Delta_{t+1} Z - \mathbb{P}_t \Delta_{t+1} Z) \\
&= \mathbb{P}_t \Delta_{t+1} - \mathbb{P}_t \mathbb{P}_t \Delta_{t+1} + M_t \text{ since } M_t \text{ is } \mathcal{F}_t \text{ measurable} \\
&= \mathbb{P}_t \delta \varepsilon_{t+1} b_{t+1} - \mathbb{P}_t \mathbb{P}_t \delta \varepsilon_{t+1} b_{t+1} + M_t \\
&= M_t \text{ (since } \mathbb{P}_t \delta \varepsilon_{t+1} b_{t+1} \text{ is } \mathcal{F}_t \text{ measurable)}
\end{aligned}$$

□

Since the choice of the first martingale value is arbitrary, let $M_0 = 0$.

The general strategy for finding a martingale bound is to find an expression for

$$\mathbb{P}\{M_{t+1} \geq \eta\} \leq \inf_{\theta \geq 0} e^{\theta \eta} \mathbb{P} e^{\theta M_{t+1}}.$$

For a martingale M_t we have

$$\begin{aligned}
\mathbb{P}\{M_{t+1} \geq \eta\} &= \mathbb{P}\{e^{\theta M_{t+1}} \geq e^{\theta \eta}\} \text{ for } \theta \geq 0 \\
&\leq \mathbb{P} e^{\theta (M_{t+1} - \eta)} \text{ (by the Markov Inequality)} \\
&= \mathbb{P} e^{\theta (M_t + \Delta_{t+1} M - \eta)} \\
&= \mathbb{P} e^{\theta (M_t - \eta)} \mathbb{P}_t e^{\theta \Delta_{t+1} M} \qquad (4.1)
\end{aligned}$$

Equation (4.1) shows that it may be possible to express the tail-bound of the martingale recursively. If the conditional Moment Generating Function $\mathbb{P}_t \exp[\theta \Delta_{t+1} M]$ can be expressed as $\exp[\theta g(\theta)]$ and $g(\theta)$ is deterministic, then

$$\begin{aligned}
\mathbb{P} e^{\theta (M_t - \eta)} \mathbb{P}_t e^{\theta \Delta_{t+1} M} &\leq \mathbb{P} e^{\theta (M_t - \eta + g(\theta))} \\
&\leq e^{\theta (t \cdot g(\theta) - \eta)}. \qquad (4.2)
\end{aligned}$$

The problem of finding a bound can now be accomplished by optimizing over values of θ.

4.2. **(Hoeffding Bound).** For a random variable with zero mean and bounded ranges, the Hoeffding Bound uses convexity to bound the moment generating function. Since $\exp[\theta \Delta_{t+1} M]$ is convex and $\Delta_{t+1} M \leq n$ we can write

$$e^{\theta \Delta_{t+1} M} \leq e^{-\theta n} \left(\frac{n - \Delta_{t+1} M}{2n} \right) + e^{\theta n} \left(\frac{\Delta_{t+1} M + n}{2n} \right).$$

Taking the conditional expectation of both sides and realizing that $\mathbb{P}_t \Delta_{t+1} M = 0$ we get

$$\mathbb{P}_t e^{\theta \Delta_{t+1} M} \leq \frac{1}{2} e^{-\theta n} + \frac{1}{2} e^{\theta n} \tag{4.3}$$

Taking the log of Equation (4.3) and letting $\alpha = 1 - \beta = 1/2$ and $u = 2\theta n$ we get

$$\log \mathbb{P}_t e^{\theta \Delta_{t+1} M} \leq \log \left(\frac{1}{2} e^{-u/2} + \frac{1}{2} e^{u/2} \right) = -\frac{u}{2} + \log \left(\frac{1}{2} + \frac{1}{2} e^{u} \right).$$

In Pollard (1984, Appendix B) it is shown that this equation can be bound using a second order Taylor approximation. Performing a similar approximations we get

$$\log \mathbb{P}_t e^{\theta \Delta_{t+1} M} \leq \frac{1}{2} \theta^2 n^2.$$

Following Inequality (4.2)

$$\mathbb{P}\{M_t \geq \eta\} \leq e^{-\eta\theta + tn^2\theta^2}. \tag{4.4}$$

Minimizing in θ

$$\theta = \frac{\eta}{2tn^2}.$$

Substituting this back into Inequality (4.4)

$$\begin{aligned} \mathbb{P}\{M_t \geq \eta\} &\leq \exp \left[-\frac{\eta}{2tn^2} + \frac{tn^2\eta^2}{4t^2n^4} \right] \\ &= \exp \left[-\frac{\eta^2}{4tn^2} \right]. \end{aligned}$$

For the sake of comparing this bound with other bounds, let

$$\phi_h := \frac{\eta}{tn}.$$

Then, the bound is

$$\mathbb{P}\{M_t \geq \eta\} \leq \exp \left[-\frac{t\phi_h^2}{4} \right]. \tag{4.5}$$

4.3. **Bennett Bound.** For a random variable with zero mean, bounded ranges, and finite variance, the Bennett Bound uses the restricted range of the second moment to bound the exponential series expansion of the moment generating function. It should be noted that in Pollard (2001, Chapter 11) a slight refinement to this approach is employed to get a slightly sharper bound. This approach will also be used in finding a bound.

As shown in Pollard (2001, Appendix C), the function

$$f(x) := 2(e^x - 1 - x)/x^2 \text{ with } f(0) = 1$$

is nonnegative and increasing over the real line. We can then write

$$\begin{aligned} e^{\theta\Delta_{t+1}M} &= 1 + \theta\Delta_{t+1}M + \frac{1}{2}\theta^2\Delta_{t+1}M^2 + f(\theta\Delta_{t+1}M) \\ &\leq 1 + \theta\Delta_{t+1}M + \frac{1}{2}\Delta_{t+1}M^2 f(\theta n) \text{ (since } M_t \leq n). \end{aligned}$$

Taking the conditional expectation of both sides and using the fact that $e^x \geq 1 + x$,

$$\begin{aligned} \mathbb{P}_t e^{\theta\Delta_{t+1}M} &\leq \mathbb{P}_t \exp\left[\frac{\theta^2\Delta_{t+1}M^2}{2}\frac{2(e^{\theta n} - 1 - \theta n}{\theta^2 n^2}\right] \\ &\leq \mathbb{P}_t \exp\left[\frac{\Delta_{t+1}M^2}{n^2}\left(e^{\theta n} - 1 - \theta n\right)\right] \\ &\leq \exp\left[e^{\theta n} - 1 - \theta n\right] \text{ (again, since } \Delta_{t+1}M \leq n) \end{aligned}$$

Following Inequality (4.2),

$$\mathbb{P}\{M_t \geq \eta\} \leq \exp\left[-\theta\eta + t\left(e^{\theta n} - 1 - \theta n\right)\right] \tag{4.6}$$

and minimizing in θ we get

$$\theta = \frac{1}{n}\log\left(\frac{\eta}{tn} + 1\right).$$

Substituting this value into Inequality (4.6) we get

$$\mathbb{P}\{M_t \geq \eta\} \leq \exp\left[-\left(\frac{\eta}{n} + t\right)\log\left[\frac{\eta}{nt} + 1\right] + \frac{\eta}{n}\right].$$

Again, for the sake of comparison let,

$$\phi_b := \frac{\eta}{tn} + 1.$$

Then, the inequality becomes

$$\mathbb{P}\{M_t \geq \eta\} \leq \exp\left[t(-\phi_b \log[\phi_b] + \phi_b - 1)\right] \tag{4.7}$$

4.4. **Poisson Bound.** Both the Hoeffding Bound and the Bennett Bound have the disadvantage that that $\Delta_t M$ is bound by n. In practice we expect that the martingale increments will be small (of order c). Realizing this, a Poisson approximation will be used in attempt to get a sharper bound.

We can rewrite the conditional expectation as

$$\mathbb{P}_t e^{\theta \Delta_{t+1} M} = \mathbb{P}_t \exp\left[\theta(M_t + \Delta_{t+1} Z - \mathbb{P}_t \Delta_{t+1} Z)\right]$$

$$\exp\left[\theta\left(M_t + \frac{Z_t}{S_t}\frac{1}{2}\frac{c}{n}(n - S_t)\right)\right] \mathbb{P}_t \exp[\theta \Delta_{t+1} Z]. \tag{4.8}$$

Now, only looking at the conditional expectation term and using the fact that $|\Delta_{t+1} Z| \leq b_{t+1}$ we can write

$$\begin{aligned} \mathbb{P}_t e^{\theta \Delta_{t+1} Z} &\leq \mathbb{P}_t e^{\theta b_{t+1}} \\ &= \left(1 - \frac{c}{n} + \frac{c}{n} e^{\theta}\right)^{n - S_t} \\ &\leq \exp\left[\frac{c}{n}(n - S_t)\left(e^{\theta} - 1\right)\right]. \end{aligned} \tag{4.9}$$

It should be noted that the expression in Inequality (4.9) is the Moment Generating Function for a Poi$(c/n(n - S_t)$ variable. Since the MGF of a Poisson variable bounds the MGF of a Binomial we can replace b_{t+1} with a Poisson variable with appropriate parameter and find a bound. Equation (4.8) then becomes

$$\begin{aligned} \mathbb{P}_t e^{\theta \Delta_{t+1} M} &\leq \exp\left[\theta\left(M_t + \frac{Z_t}{S_t}\frac{1}{2}\frac{c}{n}(n - S_t)\right) + \frac{c}{n}(n - S_t)(e^{\theta} - 1)\right] \\ &\leq \exp\left[\theta M_t + \frac{c}{n}(n - S_t)\left(\frac{\theta}{2} + e^{\theta} - 1\right)\right] \text{ (since } Z_t/S_t \leq 1) \\ &\leq \exp\left[\theta M_t + n\left(\frac{\theta}{2} + e^{\theta} - 1\right)\right] \text{ (since } \frac{c}{n}(n - S_t) \leq n). \end{aligned}$$

Following Inequality (4.2),

$$\mathbb{P}\{M_t \geq \eta\} \leq \exp\left[-\theta\eta + tn\left(\frac{\theta}{2} + e^{\theta} - 1\right)\right]. \tag{4.10}$$

Minimizing in θ we get

$$\theta = \log\left[\frac{\eta}{tn} - \frac{1}{2}\right]$$

Substituting θ into Equation (4.10) we get

$$\mathbb{P}\{M_t \geq \eta\} \leq \exp\left[-\left(\eta - \frac{tn}{2}\right)\log\left[\frac{\eta}{tn} - \frac{1}{2}\right] - \frac{3}{2}tn + \eta\right].$$

To compare this with the other bounds, let

$$\phi_p := \frac{\eta}{tn} - \frac{1}{2}.$$

Then the bound becomes

$$\mathbb{P}\{M_t \geq \eta\} \leq \exp\left[tn\left(-\phi_p \log[\phi_p] + \phi_p - 1\right)\right] \tag{4.11}$$

5. COMPARING THE MARTINGALE BOUNDS

It may have already been noted that the domain of the respective ϕ values in the inequalities above may not be valid for all values of η we are interested in. The first subsection provides valid domains for each of the three inequalities. With this accomplished, it would then be interesting to find out for a given domain which bound is best. The second subsection provides this information based on approximations of the difference between the exponential terms of the bounds.

5.1. **The Domain of the Martingale Bounds.** ϕ_h, ϕ_b, and ϕ_p given in the bounds in the previous section have the common term

$$\phi := \frac{\eta}{tn}.$$

Since, $M_t \leq n$ it follows that $\phi \leq n$. Keeping this in mind, the domains will be found for each of the bounds as a function of ϕ. When looking at each bound, it will be assumed that $t, n \geq 1$.

In the case of the Hoeffding Bound, given in Equation (4.5),

$$\phi_h = \phi.$$

For this bound, values of η can range between 0 and n without discontinuities. Also, $0 \leq \mathbb{P}\{M_t \geq \eta\} \leq 1$ for η values in this range.

For the Bennett Bound, given in Equation (4.7),

$$\phi_b = \phi + 1.$$

This means that ϕ_b is in $[1, 2]$. It can be shown that

$$-t(\phi_b log[\phi_b] + \phi_b + 1) \leq 0 \text{ for } \phi_b \text{ in } [1, 2].$$

Therefore, for this bound, value of η can range between 0 and n without discontinuities. Also $0 \leq \mathbb{P}\{M_t \geq \eta\} \leq 1$ for η values in this range.

For the Poisson Bound, given in Equation (4.11),

$$\phi_p = \phi - \frac{1}{2}.$$

This means that at most ϕ_p may go between $[-1/2, 1/2]$. However, noticing the $\log$ term in the bound it is clear that, for the bound to be defined ϕ_p must me between $(0, 1/2]$. It can be shown that

$$tn(-\phi_p \log[\phi_p] + \phi_p - 1) \leq 0 \text{ for } \phi_p \text{ in } (0, 1/2].$$

It should be noted that, unlike the the exponential terms of the other bounds, this is an increasing function over the range of ϕ_p

5.2. **Comparing the Sharpness of the Bounds.** Now that it is known where the bounds are valid, we want to know which bound does better in a given domain. This will be done by comparing the exponential term in the bounds.

If the Bennett Bound is sharper than the Hoeffding Bound, then

$$t(-(\phi+1)\log[\phi+1]+\phi) \leq \frac{t\phi^2}{4} \tag{5.1}$$

or, if the expression is simplified

$$-\phi\log[\phi+1] - \log[\phi+1] - \frac{\phi^2}{4} + \phi \leq 0.$$

We can perform a second order Taylor Series approximation of the left side of the inequality, around 0.5 and get approximately

$$-\phi^2 - 0.1137984\phi - 0.09296515 \leq 0.$$

Since this inequality is true for all ϕ in $[0,1]$ the approximation indicates that the Bennett Bound beats the Hoeffding bound over ϕ, and an R calculation of Inequality (5.1) confirms this.

If the Poisson Bound is sharper than the Bennett Bound, then

$$tn(-(\phi-1/2)\log[\phi-1/2]+\phi-3/2) \leq t(-\phi\log[\phi+1]-log[\phi+1]+\phi).$$

Performing a R calculations, it can be shown that for $n \geq 3$, this inequality holds for ϕ in $(1/2, 1]$. Therefore, in this interval, The Poisson Bound beats the Bennett Bound.

To summarize, for ϕ in $[0, 1/2]$, the Bennett Bound performs the best, and for ϕ in $(0.5, 1]$ the Poisson Bound performs the best. It should be noted that since $\phi = \eta/(tn)$ the Bennett Bound will be the best for large values of t, ϕ will eventually become smaller than $1/2$.

6. Future Work

As stated in the introduction, the goal of this paper is to provide an analysis of a greedy coloring algorithm for random graphs with the hope of refining the bound on its threshold behavior. This paper explores the asymptotic behavior of a simplified model. The next step is to incorporate the fact that a each step of the algorithm, two-color stack counts are not only receiving items, but also losing items. With this accomplished, the case where vertices have 1 possible graph coloring can be incorporated. Finally, the case where the graph algorithm fails can be analyzed

The model analyzed in this paper falls under a more general class of problems where is stochastic process is modeled as a differential equation plus a random expression. These problems also provide a future direction for exploration.

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