1. A Note on Notation

It should be noted that in this paper, the notation $S_2(t)$ is used to denote the size of the 2-stacks at time t. This is different than previous papers were S_t was used to denote the number of items not in a 3-stack. Also, the notation $U_2(t)$ is used to denote the sum of the size of the fail-stack, 1-stack, and 2-stack.

2. One-Stacks Stay Small Until $n^{1/3}$

For a vertex to be a member of a 1-stack at time t, it must have been chosen twice to have a color from its color list removed. Each of the other $t-2$ times it was not selected to be colored and was not adjacent to a vertex being colored. This means that the probability of a given vertex being in a 1-stack at time t is at least

$$p_1(t) = \binom{t}{2}\left(\frac{c}{n}\right)^2\left(1-\frac{c}{n}\right)^{t-2}.$$

This implies that the number of 1-stack vertices up to time m is stochastically dominated by

$$\sum_{t=1}^{m} np_1(t) \leq \sum_{t=1}^{m} \frac{t^2}{2}\frac{c^2}{n} = \frac{c^2}{6n}m^3 + o(n)$$

Therefore, for large enough n, up to time $n^{1/3}$, the probability that the 1-stack size is greater than zero is small.

3. An Upper Bound on the Probability $U_2(t) = 0$

The probability of $U_2(t) = 0$ is the same as the probability that all uncolored vertices are in 3-stacks. For a vertex to be a member of a 3-stack at time t, it must have never been chosen to be colored and it must not be next to a colored vertex. Therefore, the probability of a given vertex being in a 3-stack at time t is

$$p_3(t) = (1-c/n)^t$$

and, $S_3(t)$ is Bin$(n-t, (1-c/n)^t)$. Then the probability all vertices are in 3-stacks is

$$\mathbb{P}\{S_3(t) = n-t\} = (1-c/n)^{t(n-t)} \leq \exp\left(-\frac{c}{n}t(n-t)\right).$$

It may be noted that for $t \leq n$ the exponent term is strictly decreasing. If the range is restricted to $0 \leq \tilde{t} \leq n/2$ then

$$\mathbb{P}\{S_3(\tilde{t}) = n-\tilde{t}\} =\leq \exp(-c\tilde{t})$$

This implies that for some $\alpha > 0$ if we would like to control the probability

$$\mathbb{P}\{S_3(\tilde{t}) = n-\tilde{t}\} < \alpha$$

then we must go out at least $-1/c\log(\alpha)$ time steps.

4. An Upper Bound on the Maximum $S_2(t)$

For t where $1 \leq t \leq n\varepsilon$, the number of 3-color vertices that become 2-color vertices is stochastically dominated by

$$b_t \sim \text{Bin}(n, c/n).$$

This implies

$$\max_{1\leq t\leq n\varepsilon} S_2(t) \leq \sum_{t=1}^{n\varepsilon} b_t$$

and therefore to get an upper bound on the biggest $S_2(t)$ it is sufficient to get a bound on the size of the sum of b_t. This can be done by centering each of the b_t values and then applying the Bennett Inequality

$$\begin{aligned} \mathbb{P}\left\{\sum_{t=1}^{n\varepsilon} b_t \geq \nu\right\} &= \mathbb{P}\left\{\sum_{t=1}^{n\varepsilon} b_t - n\varepsilon c \geq \nu - n\varepsilon c\right\} \\ &\leq \exp\left(-\frac{(\nu - n\varepsilon c)^2}{n\varepsilon c(1-c/n)}\psi\left(\frac{(\nu - n\varepsilon c)}{\varepsilon c(1-c/n)}\right)\right) \end{aligned}$$

where

$$\psi(x) = 1 + x^{-1} + x^{-2}\log(x+1).$$

As ν gets larger than the expected value of the sum of b_t's, that is $\nu > n\varepsilon c$, there is a sharp decrease in the probability of getting an $S_2(t)$ larger than ν.