Linear Models Homework 2 Solution to Problem 2

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Let P_W denote the projection matrix for the subspace spanned by W. Then

$$\widehat{Y} = P_W Y = W\alpha \in \text{sp}(W)$$

and

$$Y = \widehat{Y} + (I - P_W)Y = W\alpha + \widetilde{Y}. \tag{1}$$

Where $\widetilde{Y} \in \text{sp}(W)^\perp$.

For each i,

$$\widehat{x}_i = P_W x_i \ \in \text{sp}(W)$$

and $x_i - \widehat{x}_i \in \text{sp}(W)^\perp$. That is, each $\widehat{x}_i$ is a linear combination of columns of W. Thus,

$$\widehat{X} = WB$$

for some $s \times p$ matrix B and $X = WB + \widetilde{X}$.

Similarly

$$P_{\widetilde{X}}\widetilde{Y} = \widetilde{X}\gamma$$

and

$$\widetilde{Y} = \widetilde{X}\gamma \qquad \text{where} \qquad R = (I - P_{\widetilde{X}})\widetilde{Y} \tag{2}$$

(i) By Equation (2), R must be orthogonal to $\text{sp}(\widetilde{X})$. We can also rewrite this equation as

$$R = \widetilde{Y} - \widetilde{X}\gamma, \tag{3}$$

a linear combination of vectors orthogonal to $\text{sp}(W)$. Thus $R \perp \text{sp}(W)$.

We still need to show $R \perp \text{sp}(X)$. Every vector z in $\text{sp}(M)$ can be written as a linear combination,

$$z = Wc + Xd = W(c + Bd) + \widetilde{X}d$$

for some $c \in \mathbb{R}^s$ and $d \in \mathbb{R}^p$. From Equation (1 and 2) we know that $R'W = 0$ and $R'\widetilde{X} = 0$. It follows that $R \perp \mathrm{sp}(M)$.

(ii) Using Equation (1) and the results from above we can write

$$\begin{aligned} Y - R &= W\alpha + \widetilde{Y} - (\widetilde{Y} - \widetilde{X}\gamma) \\ &= W\alpha - \widetilde{X}\gamma \\ &= W(\alpha + B\gamma) - X\gamma \ \in \mathrm{sp}(M). \end{aligned}$$

Y is a linear combination of elements from W and X which is in $\mathrm{sp}(M)$.

(iii) - (iv) By uniqueness of the orthogonal decomposition of Y, it follows that

$$Y - R = P_M Y \qquad \text{and} \qquad R = (I - P_M)Y.$$

That is,

$$Y - R = W(\alpha + B\gamma) - X\gamma = M \begin{bmatrix} \alpha + B\gamma \\ -\gamma \end{bmatrix}$$

is the projection of Y onto $\mathrm{sp}(M)$.