Linear Models Homework 2 Solution to Problem 2

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Let P_W denote the projection matrix for the subspace spanned by W, then

$$\hat{Y} = P_W Y = W\alpha \in \text{sp}(W).$$

In other words, the estimates of Y are a linear combination of columns of W. Then we can write

$$\tilde{Y} = (1 - P_W)Y \in \text{sp}(W^{\perp}). \tag{1}$$

$\tilde{Y}$ is the vector of residuals from regressing W onto Y. Y can then be expressed as

$$Y = \hat{Y} + \tilde{Y} \tag{2}$$

Proceeding similarly we get:

$$\begin{aligned} \hat{X} &= P_W X = W\beta \in \text{sp}(W), \\ \tilde{X} &= (1 - P_W)X \in \text{sp}(W^{\perp}), \\ X &= \hat{X} + \tilde{X}. \end{aligned} \tag{3}$$

Let Z be $\tilde{Y}$ projected onto the $\tilde{X}$ space

$$Z = P_{\tilde{X}}\tilde{Y}.$$

Then we can write

$$\begin{aligned} \tilde{Y} &= Z + R \\ R &= (1 - P_{\tilde{X}})\tilde{Y} \end{aligned} \tag{4}$$

(i) By Equation (4), R is $\tilde{Y}$ projected onto the subspace orthogonal to $P_{\tilde{X}}$. R must be orthogonal to $\text{sp}(\tilde{X})$. We can also rewrite this equation as

$$R = \tilde{Y} - P_{\tilde{X}}\tilde{Y} = \tilde{Y} - \tilde{X}\gamma. \tag{5}$$

R is $\tilde{Y}$ minus a linear combination of columns in in $\tilde{X}$. From Equations (1) and (3), we know that both $\tilde{X}$ and $\tilde{Y}$ are in $\text{sp}(W^{\perp})$. Therefore, R must also be in $\text{sp}(W^{\perp})$.

We still need to show R is orthogonal to $\text{sp}(X)$. For any $x \in \text{sp}(X)$, there exists a v_1 and v_2 such that

$$x = W\beta v_1 + \tilde{X}v_2.$$

Taking the projection of R onto a vector x,

$$\langle R, x\rangle = \langle R, W\beta v_1\rangle + \langle R, \tilde{X}v_1\rangle. \tag{6}$$

Since R is orthogonal to $\text{sp}(\tilde{X})$, the second term is zero. Using Equation (5)

$$\langle R, W\beta\rangle = \langle \tilde{Y} - \tilde{X}\gamma, W\beta v_1\rangle = \langle \tilde{Y}, W\beta v_1\rangle - \langle \tilde{X}\gamma, W\beta v_1\rangle \tag{7}$$

From Equations (1) and (3), we know that $\tilde{Y}$ and $\tilde{X}$ are orthogonal to $\text{sp}(W)$ and so the projection must be zero.

(ii) Using Equation (2) and the results from above we can write

$$\begin{aligned} Y - R &= W\alpha + \tilde{Y} - (\tilde{Y} - \tilde{X}\gamma) \\ &= W\alpha - \tilde{X}\gamma \\ &= W\alpha - X\gamma + P_W X\gamma \\ &= W(\alpha + \beta\gamma) - X\gamma. \end{aligned} \tag{8}$$

Y is a linear combination of elements from W and X which is in $\text{sp}(M)$.

(iii) - (iv) From Equation (8) we can write

$$Y - R = \hat{Y}_M = \left[\begin{array}{c} W X \end{array}\right] \left[\begin{array}{c} \alpha + \beta\gamma \\ -\gamma \end{array}\right] \tag{9}$$

Y can be represented as Y projected onto $\text{sp}(M)$ plus Y projected onto the subspace orthogonal to $\text{sp}(M)$ which is R.