Risk, Ambiguity and the Ellsberg Paradox

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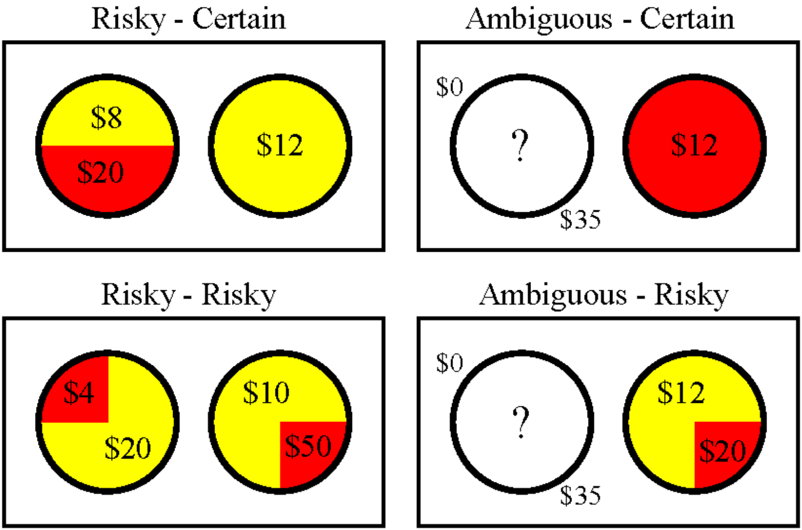
Neural Signatures of Preferences for Risk and Ambiguity

A **certain option** is a lottery where the rewards are equal in every state of the world.
A **risky option** is a lottery where the relative liklihoods of rewards are known.

An **ambiguous option** is a lottery where the relative liklihoods of rewards are unknown.

Figures


Risky - Certain
$8
$20
$12
Ambiguous - Certain
$0
?
$35
$12
Risky - Risky
$4
$20
$10
$50
Ambiguous - Risky
$0
?
$35
$12
$20

Preferences for Risk

Theory of Games and Economic Behavior, von Neumann and Morgenstern (1944):

Expected Utility:

$$E^{\beta}_{\pi}(x) \equiv \pi_1 u(x_1) + \pi_1 u(x_2) = \pi_1 (x_1)^{\beta} + \pi_1 (x_2)^{\beta}$$

Estimate β in $E^{\beta}_{\eta}(\cdot)$ by fitting

$$\arg\max_{x,y} \{E^{\beta}_{\pi}(x), E^{\beta}_{\lambda}(y)\}$$

to choices between pairs of risky options: $[x; \pi]$ and $[y; \lambda]$.
Values of $\widehat{\beta} > 1$ indicate risk-seeking and values of $\widehat{\beta} < 1$ indicate risk-aversion.

Preferences for Ambiguity

"Differential Ambiguity and Ambiguity Attitude," Ghirardato et al. (2004)
$\alpha - \max\min$ **Expected Utility**:

$$W_\alpha^\beta(z) \equiv (1-\alpha)((z_1)^\beta) + \alpha((z_2)^\beta)$$

Estimate α in $W_\alpha^{\widehat{\beta}}(\cdot)$ by fitting

$$\arg\max_{y,z}\{E_\lambda(y), W_\alpha^{\hat{\beta}}(z)\}$$

to choices between pairs of risky and ambiguous options: $[y;\lambda]$ and z.
Values of $\widehat{\alpha} < 0.5$ indicate ambiguity-seeking and values of $\widehat{\alpha} > 0.5$ indicate ambiguity-aversion.

The Neural Signatures Model

Subjects evaluate risky options $[x;\pi]$ with the expected utility function of von Neumann and Morgenstern:

$$E_{\pi}^{\beta}(x).$$

Subjects evaluate ambiguous options z with the α-maxmin expected utility function, of Ghirardato et al:

$$W_{\alpha}^{\beta}(z)$$

In making choices between two options, subjects choose the option with the higher valuation.

Predicting Choice Behavior

Values of $\widehat{\alpha}$ and $\widehat{\beta}$ were uncorrelated across the 13 subjects (nine male;$18 - 33$ years), indicating that risk and ambiguity made independent cotributions to the choice process. Estimated parameters accurately predicted the subjects choices, i.e.,an average of 75% in choices under risk with a minimum of 68% and a maximum of 86% and an average of 79% in choices under ambiguity with a minimum of 71% and a maximum of 91%. "

Predicting Neural Activity

Given the correlations between the estimates $(\hat{\alpha}, \hat{\beta})$ and *fMRI* measures of neural activity in different regions of the brain, Huetell et al. (2006) conclude that "decision making under ambiguity does not represent a special, more complex case of risky decision making; instead these two forms of uncertainty are supported by distinct mechanisms."

Preferences for Ambiguity with Contingent Beliefs

The *ADM* model of Bracha and Brown (2009) is a model of ambiguity-seeking behavior. The variational preferences model of Maccheroni et al. (2006) is a model of ambiguity-averse behavior.The defining property of these models is that subjects choose their beliefs as a function of the distribution of rewards, i.e.,"contingent beliefs"

Schmeidler's Axiom

MMR require the preference order over ambiguous options or simple acts to satisfy six axioms. The fifth axiom is "Schmeidler's axiom"on the shape of indifference curves.

(i) quasi-concave: variational preferences (ambiguity-averse)
(ii) quasi-linear: *SEU* preferences (ambiguity-neutral)
(iii) quasi-convex: *ADM* preferences (ambiguity-seeking)

Representation Theorems (1)

If $z \equiv (z_1, z_2)$ and $U(z) \equiv (u(z_1), u(z_2))$ then the *ADM* model and the variational preferences model can be represented as composite utility functions

$$V(z) \equiv J(U(z)).$$

Let $\Delta \equiv \{\pi \in R^2_+ : \pi_1 + \pi_2 = 1\}$, where $J^*(\pi)$ is the Legendre-Fenchel conjugate of $J(U(z))$.In the *ADM* representation, $J(U(z))$ is convex in

$U(z)$ and

$$J(U(z)) = \max_{\pi \in \Delta}\{\langle U(z), \pi\rangle - J^*(\pi)\},$$

where

$$\nabla_{U(z)} J(U(z)) = \arg\max_{\pi \in \Delta}\{\langle U(z), \pi\rangle - J^*(\pi)\}$$

are the "contingent beliefs".

Representation Theorems (2)

In the variational preferences representation, $J(U(z))$ is concave in $U(z)$ and

$$J(U(z)) = \min_{\pi \in \Delta} \{ \langle U(z), \pi \rangle - J^*(\pi) \}$$

where

$$\nabla_{U(z)} J(U(z)) = \arg\max_{\pi \in \Delta} \{ \langle U(z), \pi \rangle - J^*(\pi) \}$$

are the "contingent beliefs".

The Log-Partition Function

The log-partition function $J(U(z)) \equiv \lg[\eta_1 \exp u(z_1) + \eta_2 \exp u(z_2)]$ is a parametric example of *ADM* or ambiguity-seeking preferences, where $\nabla_{U(z)} J(U(z)) \in \Delta$. The Legendre-Fenchel conjugate of the log-partition function is the relative entropy function or Kullback-Leibler divergence $J^*(\pi) \equiv [\pi_1 \lg(\pi_1/\eta_1) + \pi_2 \lg(\pi_2/\eta_2)]$.

Multiplier Preferences

Hansen and Sargent (2001) introduced multiplier preferences $J(U(z)) \equiv -\ln[\eta_1 \exp -u(z_1) + \eta_2 \exp -u(z_2)]$, where $\nabla_{U(z)} J(U(z)) \in \Delta$, defined as the Legendre-Fenchel conjugate of the negative relative entropy function. Strzalecki (2008) gives an axiomatic characterization of multiplier preferences, where he adds two of Savage's axioms to the axioms for variational preferences. As such they are a parametric example of ambiguity-averse preferences.

The Neural Signatures Model with Contingent Beliefs

Subjects evaluate risky options, $[x;\pi]$, by computing the expected value

$$E_\pi(x) = \pi_1 u(x_1) + \pi_2 u(x_2).$$

Subjects evaluate ambiguous options, z, by computing

$$J(U(z)) = \max_{\pi\in\Delta}\{\langle U(z),\pi\rangle - J^*(\pi)\}$$

or

$$J(U(z)) = \min_{\pi\in\Delta}\{\langle U(z),\pi\rangle - J^*(\pi)\}$$

as defined in Bracha and Brown (2009) and in Maccheroni et al. (2006). If offered the choice between two options, subjects chooses the option with the higher valuation.

Risk, Ambiguity and the Savage Axioms.

Ellsberg (1961) proposed a thought experiment, where subjects are asked to bet on a draw from a risky urn, containing 50 black and 50 white balls or bet on a draw from an ambiguous urn, containing 100 balls, where the proportion of black balls is unknown.In each case, the ambiguous option and the risky option define the same state-contingent claim, i.e., they have the same distribution of rewards.

States of the World

The White (Black) Ball Wins

Suppose the announced color is white (black), then the subject's choice is between an ambiguous option where the rewards are \$100, if she draws a white(black) ball and \$0 if she draws a black (white) ball and a risky option with the same distribution of rewards and relative likelihoods of 1/2.

Subjects in Group A satisfy Savage's axioms for SEU (Ambiguity-Neutral). Subjects in Group B are indifferent. Subjects in Group C always choose the ambiguous urn (Ambiguity-Seeking).Subjects in Group D always choose the risky urn (Ambiguity-Averse).The choice behavior of groups C and D constitute the Ellsberg paradox

Ellsberg's Explanation of the Paradox

For Groups C and D,$\cdots$ "we would have to regard the subject's subjective probabilities as being dependent upon his pay-offs, his evaluation of the outcomes"$\cdots$ it is impossible to infer from the resulting behavior a set of probabilities for events independent of his payoffs."

Explaining the Ellsberg Paradox with Contingent Beliefs (1)

Suppose $z = (100, 0)$ and $w = (0, 100)$ are ambiguous options where $[\pi, z]$ and $[w, \lambda]$ are the corresponding risky options. Let

$$J(U(z)) > E_{\pi}(z) \text{ and } J(U(w)) > E_{\pi}(w)$$

If $J(U(z))$ is the log-partition function and $U(0) = 0$,, then

$$J(U(0)) = 0$$

and $J(U(t))$ is convex in $U(t)$, i.e., the subject is ambiguity-seeking. Hence for all ambiguous options t:

$$J(U(t)) \leq [\nabla_{U(t)} J(U(t)) \cdot U(t)].$$

follows from the first order condition:

$$J(U(r)) \geq J(U(s)) + [\nabla_{U(s)} J(U(s)) \cdot (U(r) - U(s))]$$

for all ambiguous options r and s.

Explaining the Ellsberg Paradox with Contingent Beliefs (2)

If

$$J(U(z)) > E_\pi(z) \text{ and } J(U(w)) > E_\pi(w)$$

then

$$E_\pi(w) < [\nabla_{U(w)} J(U(w)) \cdot U(w)]$$

and

$$E_\pi(z) < [\nabla_{U(z)} J(U(z)) \cdot U(z)]$$

where

$$\nabla_{U(w)} J(U(w)) \in \Delta \text{ and } \nabla_{U(z)} J(U(z)) \in \Delta$$

Explaining the Ellsberg Paradox with Contingent Beliefs (3)

If

$$J(U(w)) < [\nabla_{U(x)} J(U(w)) \cdot U(w)]$$

and

$$J(U(z)) < [\nabla_{U(z)} J(U(z)) \cdot U(z)]$$

then

$$0 < [\nabla_{U(w)} J(U(w)) - \pi] \cdot U(w).$$

and

$$0 < [\nabla_{U(z)} J(U(z)) - \pi] \cdot U(z).$$

The ambiguity-seekng subjects in Group C "act as if" the relative likelihood of a positive reward is greater in the ambiguous urn than in the risky urn, consistent with Ellsberg's explanation.

A Measure of Ambiguity

If $z = (z_1, z_2)$ is an ambiguous option and there exists relative likelihoods π_1^* and π_2^* such that

$$E_{\pi^*}(U(z)) \equiv \pi_1^* u(z_1) + \pi_2^* u(z_2) = J(u(z_1), u(z_2)) \equiv J(U(z)),$$

then we define the unique, risky option $[z; \pi^*]$ as the *risky equivalent* of the ambiguous option z.If $[x; \pi^*]$ is the risky equivalent for an ambiguous option x, then we suggest $H(\pi^*) = -[\pi_1^* \ln \pi_1^* + \pi_2^* \ln \pi_2^*]$, the entropy of π^*, as a precise measure of the ambiguity of z.
This suggestion is consistent with the information-theoretic definition of entropy as a measure of " missing information" or the amount of uncertainty (ambiguity).

A Measure of Ambiguity: Examples

(i) If $J(U(z))$ is homogeneous of degree 1, then $[\nabla_{U(z)} J(U(z)) \cdot U(z)] = J(U(z))$.If $\nabla_{U(z)}\ J(U(z)) \in \Delta$, then$[z, \nabla_{U(z)} J(U(z))]$is the risky equivalent of z and $H(\nabla_{U(z)} J(U(z))$ is the ambiguity of z.
(ii)In the Ellsberg thought experiment, the risky urn, with the uniform distribution $\pi = (\frac{1}{2}, \frac{1}{2})$, is the risky equivalent of the ambiguous urn for the subjects in Group B. Hence the entropy of the ambiguous urn, $H(\frac{1}{2}, \frac{1}{2})$, is maximal.

A Parametric Family of Contingent Beliefs Model

Let

$$J^{\beta}_{\alpha,\eta}(U(z)) \equiv \frac{1}{\alpha^3} \lg[\eta_1 \exp \alpha^3 (z_1)^{\beta} + \eta_2 \exp \alpha^3 (z_2)^{\beta}]$$

, where $\alpha \in (-1, 0) \cup (0, 1)$, β and $\delta \in (0, \infty)$ and $\eta = (\eta_1, \eta_2) \in \Delta$. $u(s) = (s)^{\gamma}$ is concave (risk-averse) for $\gamma \in (0, 1)$ and convex (risk-seeking) for $\gamma \in (1, \infty)$. $J^{\beta}_{\alpha,\eta}(U(z))$ is concave (ambiguity-averse) for $\alpha \in (-1, 0)$ and convex (ambiguity-seeking) for $\alpha \in (0, 1)$, and

$$E^{\delta}_{\pi}(U(x)) \equiv \pi_1 (x_1)^{\delta} + \pi_2 (x_2)^{\delta}$$

Estimation

(i) We estimate $\widehat{\delta}$ by fitting the expected utility model

$$E_{\pi}^{\delta}(U(x)) \equiv \pi_1(x_1)^{\delta} + \pi_2(x_2)^{\delta}$$

to choice data between pairs of risky and certain options.
(ii) We estimate $\widehat{\alpha}$ and $\widehat{\beta}$ and by fitting the parametric, contingent beliefs model

$$J_{\alpha,(\frac{1}{2},\frac{1}{2})}^{\beta}(U(z)) \equiv \frac{1}{\alpha^3} \lg[\frac{1}{2} \exp \alpha^3 (z_1)^{\beta} + \frac{1}{2} \exp \alpha^3 (z_2)^{\beta}]$$

to choice data between pairs of ambiguous options.
(iii) We test the estimated neutral signatures model with contingent beliefs on pairs of risky and ambiguous options and pairs of certain and ambiguous options, where we compute the prediction error of the estimated model. Moreover, we test if $\widehat{\delta} = \widehat{\beta}$