Optimal Estimation of Multidimensional Normal Means with Unknown Variances

Xu Han

Department of Statistics
The Wharton School
University of Pennsylvania

Joint work with Professor Lawrence D. Brown

Yale University
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The Normal Mean Estimation Problem

$$X \sim N_p(\theta, \eta^{-2} I) \qquad W \sim \eta^{-2}\chi^2_m$$

- θ and η unknown;
- W independent of X;
- $\eta = 1/\sigma$ is called precision;
- Squared Error Loss: $L(\theta, \eta^2; \delta(x, w)) = ||\delta(x, w) - \theta||^2\eta^2$;
- Risk function: $R(\theta, \eta^2; \delta(X, W)) = E_{\theta,\eta^2} L(\theta, \eta^2; \delta(X, W))$.

Use procedure $\delta(x, w)$ to estimate θ.

Bayes Estimator:

$\delta_G(x, w) = \frac{E(\theta\eta^2|x,w)}{E(\eta^2|x,w)}$ with prior G on θ, η.

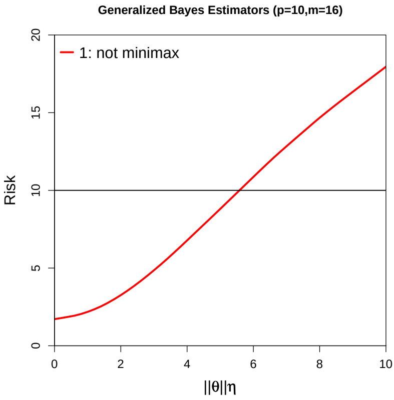
- G proper, proper Bayes (Strawderman (1973));
- G improper, generalized Bayes (Maruyama & Strawderman (2005)).

Benchmark for Evaluation of Estimators: Admissibility and Minimaxity

δ_1 dominates δ_2 if $R(\delta_1; \theta, \eta^2) \leq R(\delta_2; \theta, \eta^2)$ for all θ, η^2, and with strict $<$ for some θ, η^2.

Benchmark:

- Admissible estimator: no other estimators can dominate it;
- Minimax estimator: if it dominates MLE x.
- This talk will focus on admissible estimators.



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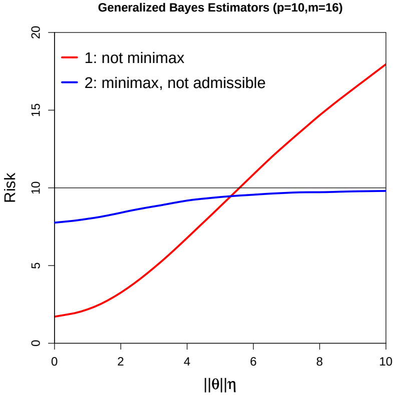

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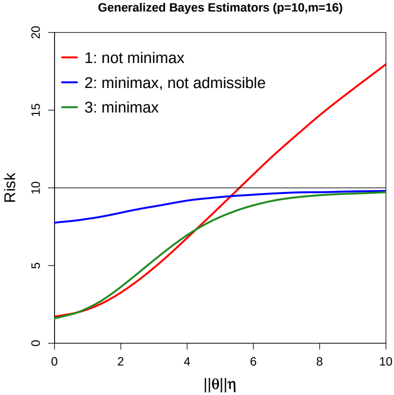
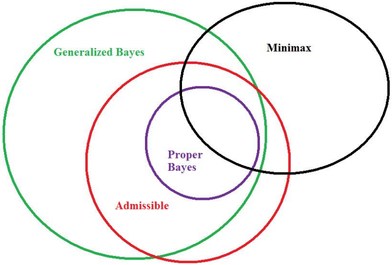


Unknown Variance Case



Admissibility:

- Strawderman (1973): proper Bayes admissible and minimax estimators for $p \geq 5$.

Minimaxity:

- Maruyama & Strawderman (2005), Wells & Zhou (2007): minimax generalized Bayes estimators for $p \geq 3$.

- Admissibility of generalized Bayes estimators is an important and open problem.
- Goal: Provide sufficient conditions for prior density functions such that δ_G is admissible.

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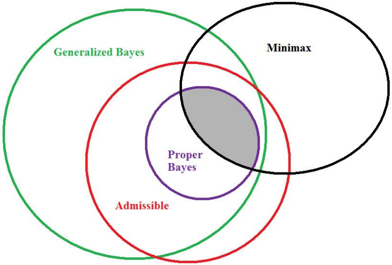

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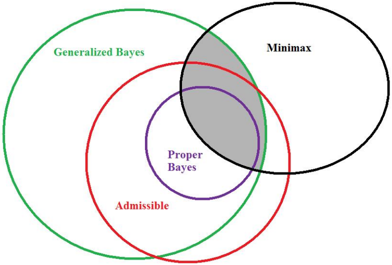

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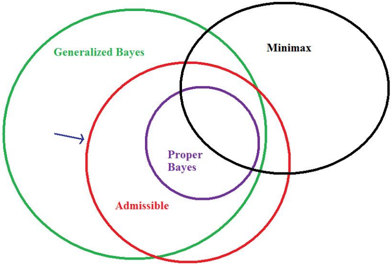

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Admissibility Theorem

Hierarchical Bayes Model:

$$\theta|\eta^2 \sim g(\theta|\eta^2) \qquad \eta^2 \sim \pi(\eta^2)$$

Admissibility Theorem:

Let $X \sim N_p(\theta, \eta^{-2} I_p), W \sim \eta^{-2}\chi^2_m$, where X and W are independent. If $g(\theta|\eta^2)$ and $\pi(\eta^2)$ satisfy Condition 1, 2 and 3, then δ_G is admissible.

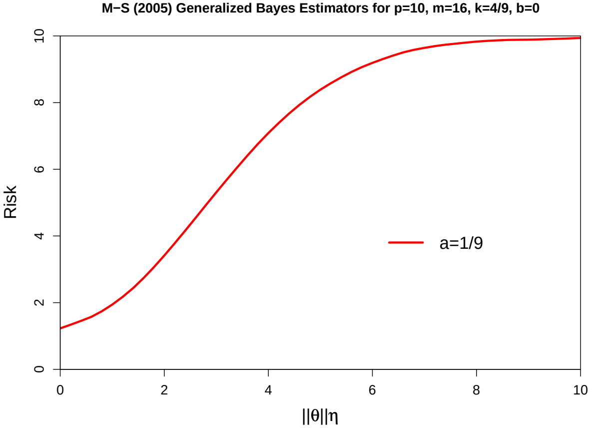
Application in Maruyama-Strawderman (2005) Hierarchical Bayes Model:

$$\begin{aligned} \theta|\nu,\eta &\sim N_p(0, \nu\eta^{-2} I) \\ h(\nu) &\propto \nu^b (1+\nu)^{-a-b-2} \qquad \nu \geq 0 \\ \pi(\eta) &\propto \eta^{-2k} \qquad \eta \geq 0 \end{aligned}$$

Corollary:

When $-a + 1/2 < k < 1/2$, and $b \geq 0$, δ_G is admissible.

Performance of Generalized Bayes Estimators



- a=1/9 admissible
- a=-2 outside admissible range
- a=-5 far outside admissible range

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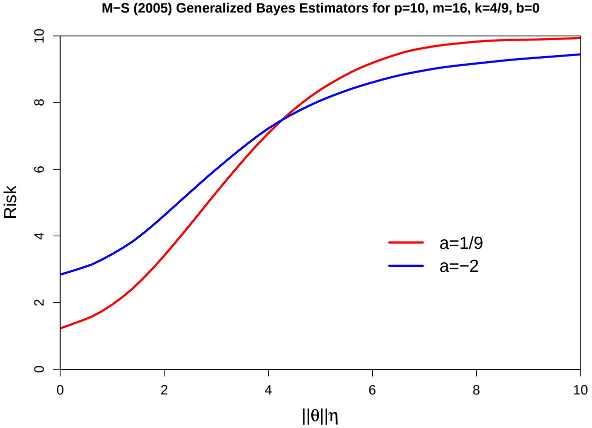

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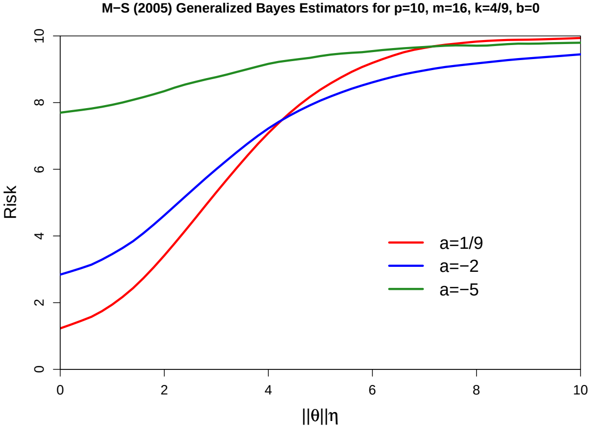

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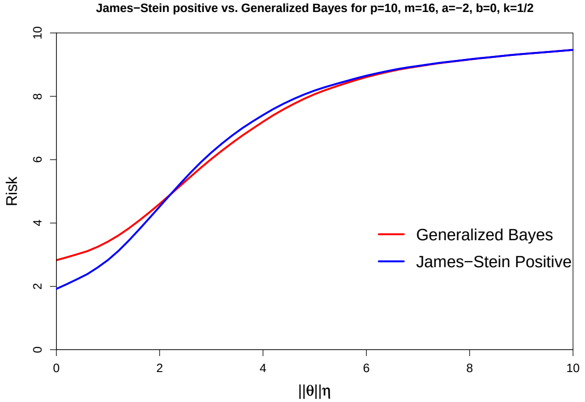
Possible Improvement

Conjecture: When $-a - 3/2 < k < 1/2, b \geq 0$, $\delta_G(x, w)$ is admissible. This is the best sufficient condition for admissibility.

- $a = -2, b = 0, k = 1/2$ are on the boundary of sufficient conditions for admissibility in conjecture.



Summary

- Admissible estimation of normal mean with unknown variances is important problem;
- Admissibility and minimaxity provide benchmark for shrinkage estimators;
- Admissibility is powerful tool to select hierarchical priors;
- Admissibility Theorem gives sufficient conditions for admissibility;
- Corollary describes a subset of M-S hierarchical Bayes estimators that are minimax and admissible.

Thank You!