Functional Generalized Linear Models and Applications

Wei Dou

Department of Statistics, Yale University

Jointly with
David Pollard and Harrison H. Zhou

May 15, 2009

The Model

Assume data pairs $(Y_i, \mathbb{X}_i(t))$ for $i = 1, 2, \cdots, n$ are i.i.d. with

$$Y_i | \mathbb{X}_i(t) \sim \exp\left\{yL_i(\mathcal{X}) - \psi(L_i(\mathcal{X}))\right\}, \qquad (1)$$

and

$$L_i(\mathcal{X}) = a + \int_{\mathcal{T}} \mathbb{B}(t)\mathbb{X}_i(t)dt$$

- Explanatory variables $\mathbb{X}_i(t)$'s are Gaussian processes and observed
- $\mathbb{K}(s, t)$ is the unknown covariance kernel of $\mathbb{X}_i(t)$'s

The Model

(Cont'd)

Denote $\mathcal{F}$ the space of parameter $(\mu, \mathbb{K}, a, \mathbb{B})$ of the model such that

- The decay of the eigenvalues:

$$\theta_{j,\mathbb{K}} \leq C_1 \cdot j^{-\alpha}$$

- The smoothness of slope $\mathbb{B}(t)$:

$$|b_j| \leq C_2 \cdot j^{-\beta} \quad \text{where} \quad b_j = \langle \mathbb{B}, \phi_{j,\mathbb{K}} \rangle$$

- Other regularity conditions $\cdots$

Asymptotic Equivalence

Theorem 1:

- If $m \asymp n^{1/(\alpha+2\beta-1-\delta)}$ $(0 < \delta < 2\beta - \alpha - 3)$
- then under some additional assumptions on α and β, we have

$$\sup_{F \in \mathcal{F}} ||\mathbb{P}_{F,n} - \tilde{\mathbb{P}}_{F,n}||_{TV} \to 0, \quad \text{as} \quad n \to +\infty$$

where

- $\mathbb{P}_{F,n}$ denotes the joint probability distribution for the full model
- $\tilde{\mathbb{P}}_{F,n}$ denotes the joint probability distribution for the simplified model with the first m principal components.

Minimax Optimal

Theorem 2:

- If we choose
  - $m \asymp n^{1/(\alpha+2\beta-1-\delta)}$ $(0 < \delta < 2\beta - \alpha - 3)$
  - $m_0 \asymp n^{1/(\alpha+2\beta)}$ and $m_0 \leq m$.
- then the M-estimator $\hat{\mathbb{B}}(t) = \sum_{j \leq m_0} \hat{b}_j \phi_{j,\hat{\mathbb{K}}}(t)$ satisfies

$$\lim_{D \to +\infty} \limsup_{n \to +\infty} \sup_{F \in \mathcal{F}} \mathbb{P}_{F,n} \left\{ ||\hat{\mathbb{B}}(t) - \mathbb{B}(t)||^2 > D n^{(1-2\beta)/(\alpha+2\beta)} \right\} = 0$$

Theorem 3:

$$\liminf_{n \to +\infty} n^{(2\beta-1)/(\alpha+2\beta)} \inf_{\breve{\mathbb{B}} \in \breve{\mathcal{B}}} \sup_{F \in \mathcal{F}} \mathbb{P}_{F,n} ||\breve{\mathbb{B}} - \mathbb{B}||^2 > 0$$

where $\breve{\mathcal{B}}$ denote the set of all measurable functions of the data.

Summary

- In order to achieve the general results, we develop a general new technique, i.e. asymptotic equivalence.
- The minimax lower bound is proved by Assouad's lemma rigorously.