Variable Selection Using the Dantzig Selector: Theoretical Properties and Extensions

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Outline

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- Adaptive (Doubly Weighted) Dantzig Selector
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Motivation: ”Omics” Data in Population-Based Studies

- The rapid advance of biotechnology yields massive high-throughput ”omics” data.
- Examples include genomics, epigenomics, proteomics, metabolomics, $\cdots$, X-omics.
- Such ”omics” technology has been applied rapidly to population-based studies to study interplay of gene and environment in causing human diseases.

Motivation: ”Omics” Data in Population-Based Studies

- In genome-wide assocation studies, a million SNP markers are genotyped across the genome to study the association of common genetic variants and disease phenotypes (case/control status).
- The massive whole genome-wide sequencing data are rapidly available, e.g., the 1000 genome project, to study the effects of rare variants.

Motivation

- Variable selection is of significant interests in current biomedical ”omics” studies.
- Common variable selection approaches are penalized likelihood based, e.g., LASSO, adaptive LASSO and SCAD.
- The Dantzig selector (DS; Candes and Tao, 2007) can be viewed as an estimating equation based variable selection procedure and is particularly appealing for longitudinal data.
- Little is known about the asymptotic properties of the DS.
- Focus of this talk (Independent data):

  (1) Large sample properties of the DS. (2) Propose Adaptive DS. (3) Connection with Adaptive Lasso.

Example: GWAS on Childhood Neuro-development in Mexico

- $n = 1000$ newborns
- 620,000 SNP markers (0,1,2) across the genome
- Outcome: Baylor score of neurodevelopment at 12 months .
- Interested in studying which genes are associated with early childhood neuro-development and gene-metal exposure interactions.

Model Setup

Model:

$$y_i = X_i^T \beta + \epsilon_i,$$

where X_i=covariates ($p \times 1$) and β=regression coefficients ($p \times 1$) and $E(\epsilon_i) = 0$ and $E(\epsilon_i^2) = \sigma^2$.

Problem:

- Suppose the true set of non-zero βs: $T^* = \{j : \beta_j \neq 0\}$.
- Goal: Identify T^* and estimate $\beta_T = \{\beta_j\}_{j \in T^*}$.

Penalized Likelihoods and Lasso

- Penalized likelihood: Simultaneous model selection and estimation.
  - Maximize

$$-\log(\text{likelihood}) + \text{sparseness penalty},$$

- Lasso

$$\min_{\beta} \frac{1}{2}||y - X\beta||_2^2 + \lambda||\beta||_1. \qquad \text{(lasso)}$$

Dantzig Selector

- Idea: Rather than controlling the size of residuals, the Dantzig selector is based on the normal score equations and controls the correlation of residuals with X:

$$\begin{array}{ll} \text{minimize} & ||\beta||_1 \\ \text{subject to} & ||X'(y - X\beta)||_\infty \leq \lambda. \end{array} \qquad \text{(DS)}$$

- Candes and Tao (2007) and Bickel al (2008) studied finite sample properties.
- The Dantzig selector and Lasso are closely related (Efron *et al.*, 2007; James, *et al.*, 2008).
- Little is known about the large sample properties of the DS.

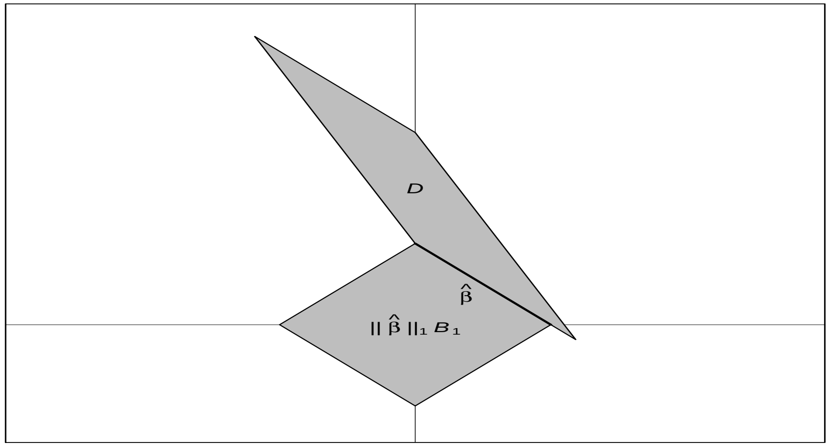
Questions

- Question(s) 1:
  Is the Dantzig selector consistent for estimation? What is its asymptotic distribution?
- Question 2:
  Is the Dantzig selector consistent for model selection? *i.e.*, is

$$\lim_{n \to \infty} P(\hat{T} = T^*) = 1?$$

- Answers to Questions 1 and 2 depend heavily on choice of λ.
- When does the Dantzig selector have a unique solution?

Uniqueness and the Dantzig Selector



- The DS feasible set: $D = \{\beta;\ ||X'(y - X\beta)||_\infty \leq \lambda\}$
- If D is *not* parallel to the closed L_1-unit ball, B_1, then the Dantzig selector has a unique solution.
- If X_i are iid then the Dantzig selector has a unique solution with probability 1.

Asymptotic Properties of the Dantzig Selector

- If $C = \lim_{n\to\infty} n^{-1}X^TX$ is *not* parallel to the L_1-ball.
- Asymptotic Limit: If $\lambda/n \to c_0$ and $c_0 \in [0, \infty]$, then $\hat{\beta} \xrightarrow{P} \beta^*$, where β^* is the true value of β and β_0 solves

$$\begin{array}{ll} \text{minimize} & ||\beta||_1 \\ \text{subject to} & ||C(\beta^* - \beta)||_\infty \le c_0. \end{array}$$

Asymptotic Properties of the Dantzig Selector

- If $\lambda/\sqrt{n} \to c_1$ and $c_1 \in [0, \infty)$, then $\sqrt{n}(\hat{\beta} - \beta^*) \xrightarrow{D} u_0$, where u_0 solves

$$\begin{array}{ll} \text{minimize} & \sum_{j \in T^*} \text{sgn}(\beta_j^*) u_j + \sum_{j \notin T^*} |u_j| \\ \text{subject to} & ||C(v-u)||_\infty \leq c_1, \end{array}$$

  and $v \sim N(0, \sigma^2 C^{-1})$.

- Model Selection Inconsistency: There exist a large class of matrices C for which the Dantzig selector is not consistent for model selection, regardless of λ.

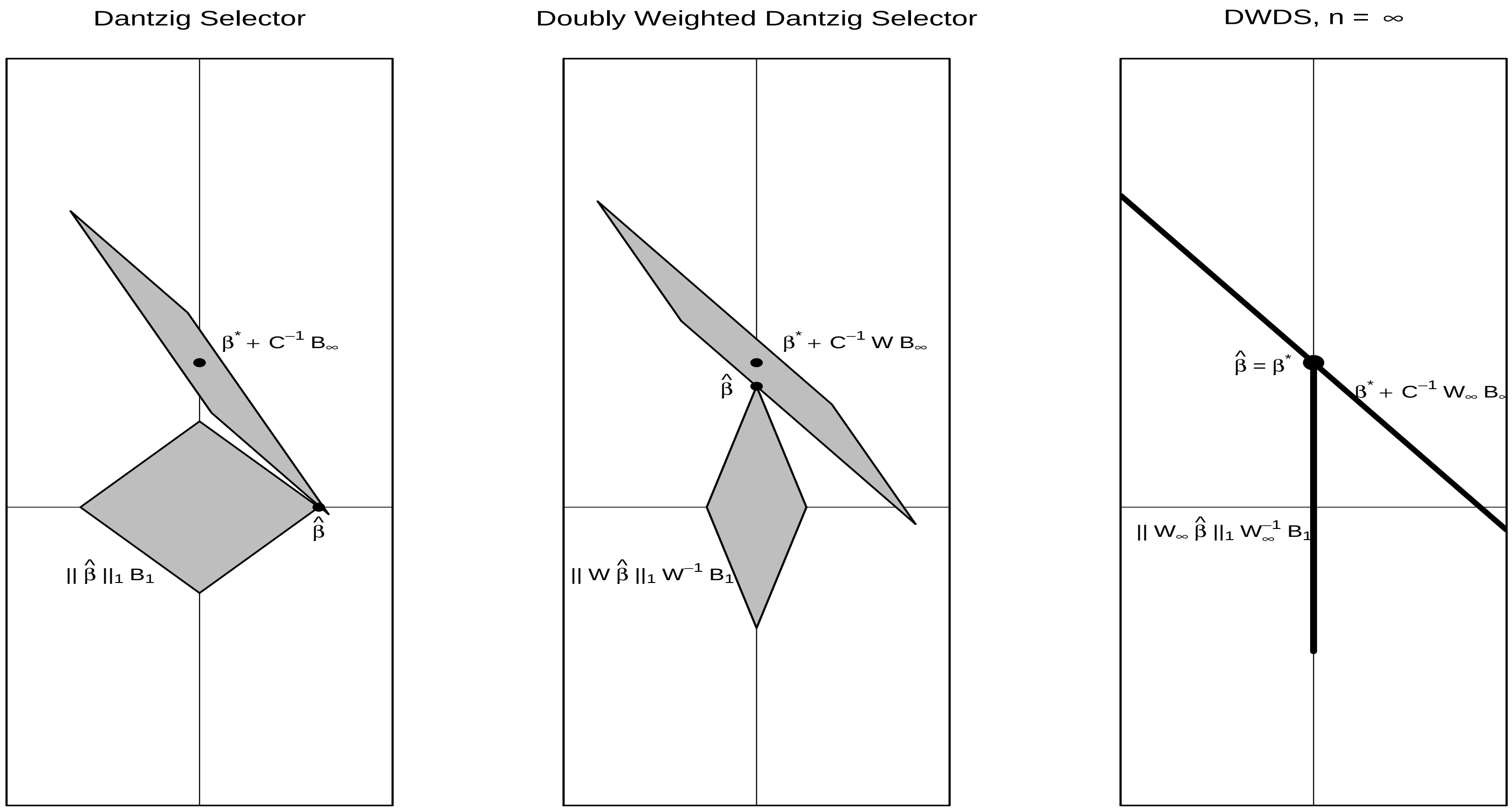
Adaptive (Doubly Weighted) Dantzig Selector

- Ideas:
  - When we suspect $\beta_j = 0$, relax the constraint on the j-th score equation and heavily penalize non-zero β_j.
  - When we suspect $\beta_j \neq 0$, Nearly solve the j-th scoring equation and only moderately penalize non-zero β_j
- Adaptively Dantzig Selector (ADS)

$$\begin{aligned} &\text{minimize} && \textstyle\sum_{j=1}^{n} w_j|\beta_j| \\ &\text{subject to} && |X_j'(y - X\beta)| \leq w_j\lambda,\ j = 1, \ldots, p. \end{aligned} \qquad \text{(ADS)}$$

e.g., $w_j = |\beta_j^{LS}|^{-\gamma}$ for some $\gamma > 0$.

Adaptive Dantzig Selector



Weighted Dantzig Selector: Asymptotics

- Suppose that $\sqrt{n}w_j/\lambda = O_P(1)$ if $j \in T^*$ and $\sqrt{n}w_j/\lambda \to \infty$ if $j \notin T^*$.
- Model Selection Consistency: The adaptive Dantzig selector(ADS) is consistent for model selection:

$$\lim_{n\to\infty} P(\hat{T} = T^*) = 1,$$

- Orcale Properties: The ADS estimators are asymptotically equivalent to the OLS estimator of β^* based on the true model T^*:

$$\sqrt{n}(\hat{\beta}_{T^*} - \beta^*_{T^*}) \xrightarrow{D} N(0, \sigma^2 C^{-1}_{T^*,T^*}).$$

Adaptive Dantzig Selector and Adaptive Lasso

- Adaptive LASSO:

$$\min_{\beta} \frac{1}{2}||y - X\beta||_2^2 + \lambda \sum_{j=1}^{p} w_j|\beta_j|. \qquad \text{(alasso)}$$

- Adaptive DS and Adaptive LASSO have the same asymptotic properties.

Adaptive Dantzig Selector and Adaptive Lasso

- Set $\beta^0 = W\beta$ and $X_0 = XW^{-1}$, we have

$$\begin{array}{ll} \text{minimize} & ||\beta^0||_1 \\ \text{subject to} & ||X_0'(y - X_0\beta^0)||_\infty \leq \lambda \end{array} \qquad \text{(ADS)}$$

and

$$\min_{\beta^0} \frac{1}{2}||y - X_0\beta^0||_2^2 + \lambda||\beta^0||_1 \qquad \text{(ALASSO)}$$

- This implies ADS and DS have the same computational cost and one can implement ADS using DASSO.

Estimation of the Toning Parameter λ

- Degrees of Freedom for the Danzig Selection:

$$\hat{df}(\lambda) = \mathrm{trace}\{X_T(X'_E X_T)^{-1} X_E\} = |T|,$$

where $E = \{j; |X'_j\{y - X\hat{\beta}\}| = \lambda\}$ and let $T = \{j;\ \hat{\beta}_j \neq 0\}$.

- BIC for DS and ADS:

$$\min_{\beta}(n\sigma^2)^{-1}||y - X\hat{\beta}||_2^2 + n^{-1}\log(n)\hat{df}(\lambda).$$

- If $w_j = |\hat{\beta}_j(\mathrm{OLS})|^{-1}$ and λ is chosen to minimize the BIC , then the ADS is

  (i) consistent for model selection and

  (ii) asymp. equiv to β^*_{LS} based on the true model, T^*.

Simulations

- Compare DS, adatpive DS, LASSO and Adaptive LASSO
- For adaptive DS (ADS) and Adaptive LASSO: $w_j = |\hat{\beta}_j^{\mathrm{OLS}}|^{-1}$
- Implementation:
  - LARS (Efron, *et al.*, 2004) for lasso and alasso.
  - DASSO (an extension of LARS; James, *et al.*, 2008) for Dantzig selector and ADS.
  - Both LARS and DASSO efficiently obtain estimates for *all* values of λ.
- λ chosen to minimize prediction error using validation data (Data validation (DV)) and BIC.

Simulation Settings

- Simulation settings:

$$\beta_0 = (3, 1.5, 0, 0, 2, 0, 0, 0) \in \mathbb{R}^8,$$

$$X = (x_{ij}),\ x_{ij} \sim N(0,1),\ \mathrm{corr}(x_{ij}, x_{ij'}) = 0.5^{|j-j'|},\ i \neq j,$$

  with independent rows.

- Noise levels $\sigma = 1$.
- Number of observations: $n = 50$.
- 500 runs.

Simulation Results

Tuning	Estimation	Sq. Error	Model Error	Model Size	$F+$	$F-$	Exact
DV	DS	0.17	0.12	5.69	2.69	0.00	0.07
	ADS	0.11	0.08	3.91	0.91	0.00	0.54
	LASSO	0.17	0.12	5.70	2.70	0.00	0.07
	ALASSO	0.11	0.09	4.01	1.01	0.00	0.50
BIC	DS	0.19	0.15	4.08	1.08	0.00	0.34
	ADS	0.12	0.09	3.22	0.22	0.00	0.83
	LASSO	0.18	0.14	4.10	1.10	0.00	0.34
	LASSO	0.12	0.09	3.23	0.23	0.00	0.83

Concluding Remarks

- Adaptive DS has advantages over the Dantzig selector: consistency in model selection and orcale properties, parallel adaptive LASSO.
- ADS outperforms DS in finite sample simulation studies.
- More complex extensions of the Dantzig selector are possible.

$$\text{LASSO} \longrightarrow \text{PL with penalty } p_\lambda$$

$$\text{Dantzig selector} \longrightarrow p_\lambda\text{-Dantzig selector}$$

- Implementation and theory are more difficult with p_λ-DS.
- Extensions to generalized linear models and longitudinal data using estimating equations are in progress .