On the entropy and log-concavity of compound Poisson measures

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Abstract

Motivated, in part, by the desire to develop an information-theoretic foundation for compound Poisson approximation limit theorems (analogous to the corresponding developments for the central limit theorem and for simple Poisson approximation), this work examines sufficient conditions under which the compound Poisson distribution has maximal entropy within a natural class of probability measures on the nonnegative integers. We show that the natural analog of the Poisson maximum entropy property remains valid if the measures under consideration are log-concave, but that it fails in general. A parallel maximum entropy result is established for the family of compound binomial measures. The proofs are largely based on ideas related to the semigroup approach introduced in recent work by Johnson [12] for the Poisson family. Sufficient conditions are given for compound distributions to be log-concave, and specific examples are presented illustrating all the above results.

Keywords

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1 Introduction

A particularly appealing way to state the classical central limit theorem is to say that, if $X_1, X_2, \dots$ are independent and identically distributed, continuous random variables with zero mean and unit variance, then the entropy of their normalized partial sums $S_n = \frac{1}{\sqrt{n}} \sum_{i=1}^n X_i$ increases with n to the entropy of the standard normal distribution, which is maximal among all random variables with zero mean and unit variance. More precisely, if f_n denotes the density of S_n and ϕ the standard normal density, then, as $n \to \infty$,

$$h(f_n) \uparrow h(\phi) = \sup\{h(f) \ : \ \text{densities } f \text{ with mean 0 and variance 1}\}, \tag{1}$$

where $h(f) = -\int f \log f$ denotes the differential entropy and log denotes the natural logarithm. Precise conditions under which (1) holds are given in [1][25][20]; also see [19][4][11] and the references therein, where numerous related results are stated, along with their history.

Part of the appeal of this formalization of the central limit theorem comes from its analogy to the second law of thermodynamics: The "state" (meaning the distribution) of the random variables S_n evolves monotonically, until the *maximum entropy* state, the standard normal distribution, is reached. Moreover, the introduction of information-theoretic ideas and techniques in connection with the entropy has motivated numerous related results (and their proofs), generalizing and strengthening the central limit theorem in different directions; see the references mentioned above for details.

The classical Poisson convergence limit theorems, of which the binomial-to-Poisson is the prototypical example, have also been examined under a similar light. An analogous program has been recently carried out in this case [23][14][9][18][12]. The starting point is the identification of the Poisson distribution as that which has maximal entropy within a natural class of probability measures. Perhaps the simplest way to state and prove this is along the following lines; first we make some simple definitions:

Definition 1.1 *For any parameter vector $\mathbf{p} = (p_1, p_2, \dots, p_n)$ with each $p_i \in [0,1]$, the sum of independent Bernoulli random variables $B_i \sim \mathrm{Bern}\,(p_i)$,*

$$S_n = \sum_{i=1}^n B_i,$$

is called a Bernoulli sum*, and its probability mass function is denoted by $b_{\mathbf{p}}(x) := \Pr\{S_n = x\}$, for $x = 0, 1, \dots$. Further, for each $\lambda > 0$, we define the following sets of parameter vectors:*

$$\mathcal{P}_n(\lambda) \ = \ \big\{\mathbf{p} \in [0,1]^n \ : \ p_1 + p_2 + \cdots + p_n = \lambda\big\} \quad \textit{and} \quad \mathcal{P}_\infty(\lambda) = \bigcup_{n \geq 1} \mathcal{P}_n(\lambda).$$

Shepp and Olkin [23] showed that, for fixed $n \geq 1$, the Bernoulli sum $b_{\mathbf{p}}$ which has maximal entropy among all Bernoulli sums with mean λ, is $\mathrm{Bin}(n, \lambda/n)$, the binomial with parameters n and λ/n,

$$H(\mathrm{Bin}(n, \lambda/n)) = \max\Big\{H(b_{\mathbf{p}}) \ : \ \mathbf{p} \in \mathcal{P}_n(\lambda)\Big\}, \tag{2}$$

where $H(P) = -\sum_x P(x)\log P(x)$ denotes the discrete entropy function. Noting that the binomial $\mathrm{Bin}(n,\lambda/n)$ converges to the Poisson distribution $\mathrm{Po}(\lambda)$ as $n \to \infty$, and that the classes of Bernoulli sums in (2) are nested, $\{b_{\mathbf{p}} : \mathbf{p} \in \mathcal{P}_n(\lambda)\} \subset \{b_{\mathbf{p}} : \mathbf{p} \in \mathcal{P}_{n+1}(\lambda)\}$, Harremoës [9] noticed that a simple limiting argument gives the following maximum entropy property for the Poisson distribution:

$$H(\mathrm{Po}(\lambda)) = \sup\Big\{H(b_{\mathbf{p}}) \ : \ \mathbf{p} \in \mathcal{P}_\infty(\lambda)\Big\}. \tag{3}$$

Partly motivated by the desire to provide an information-theoretic foundation for *compound Poisson limit theorems* and the more general problem of *compound Poisson approximation*, as a first step we consider the problem of generalizing the maximum entropy properties (2) and (3) to the case of *compound Poisson* distributions on $\mathbb{Z}_+$.[1] We begin with some definitions:

Definition 1.2 *Let P be an arbitrary distribution on $\mathbb{Z}_+ = \{0,1,\dots\}$, and Q a distribution on $\mathbb{N} = \{1,2,\dots\}$. The Q-*compound distribution *C_QP is the distribution of the random sum,*

$$\sum_{j=1}^{Y} X_j, \tag{4}$$

where Y has distribution P and the random variables $\{X_j\}$ are independent and identically distributed (i.i.d.) with common distribution Q and independent of Y. The distribution Q is called a compounding distribution, *and the map $P \mapsto C_QP$ is the Q-*compounding operation. *The Q-compound distribution C_QP can be explicitly written as the mixture,*

$$C_QP(x) = \sum_{y=0}^{\infty} P(y)Q^{*y}(x), \quad x \ge 0, \tag{5}$$

where $Q^{*j}(x)$ is the jth convolution power of Q and Q^{*0} is the point mass at $x = 0$.

Above and throughout the paper, the empty sum $\sum_{j=1}^{0}(\cdots)$ is taken to be zero; all random variables considered are supported on $\mathbb{Z}_+ = \{0,1,\dots\}$; and all compounding distributions Q are supported on $\mathbb{N} = \{1,2,\dots\}$.

Example 1.3 *Let Q be an arbitrary distribution on $\mathbb{N}$.*

1. *For any $0 \le p \le 1$, the* compound Bernoulli distribution $\mathrm{CBern}\,(p,Q)$ *is the distribution of the product BX, where $B \sim$ Bern(p) and $X \sim Q$ are independent. It has probability mass function C_QP, where P is the* $\mathrm{Bern}\,(p)$ *mass function, so that, $C_QP(0) = 1-p$ and $C_QP(x) = pQ(x)$ for $x \ge 1$.*

[1] Recall that the compound Poisson distributions are the only infinitely divisible distributions on $\mathbb{Z}_+$, and also they are (discrete) stable laws [24]. In the way of motivation we also recall Gnedenko and Korolev's remark that "there should be mathematical ... probabilistic models of the universal principle of non-decrease of uncertainty," and their proposal that we should "find conditions under which certain limit laws appearing in limit theorems of probability theory possess extremal entropy properties. Immediate candidates to be subjected to such analysis are, of course, stable laws ..."; see [8, pp. 211-215].

2. *A* compound Bernoulli sum *is a sum of independent compound Bernoulli random variables, all with respect to the same compounding distribution Q: Let $X_1, X_2, \ldots, X_n$ be i.i.d. with common distribution Q and $B_1, B_2, \ldots, B_n$ be independent Bern(p_i). We call,*

$$\sum_{i=1}^{n} B_i X_i \stackrel{\mathcal{D}}{=} \sum_{j=1}^{\sum_{i=1}^{n} B_i} X_j,$$

a compound Bernoulli sum*; in view of* (4)*, its distribution is $C_Q b_{\mathbf{p}}$, where $\mathbf{p} = (p_1, p_2, \ldots, p_n)$.*

3. *In the special case of a compound Bernoulli sum with all its parameters $p_i = p$ for a fixed $p \in [0, 1]$, we say that it has a* compound binomial distribution*, denoted by* $\mathrm{CBin}(n, p, Q)$*.*

4. *Let $\Pi_\lambda(x) = e^{-\lambda}\lambda^x/x!$, $x \geq 0$, denote the* $\mathrm{Po}(\lambda)$ *mass function. Then, for any $\lambda > 0$, the* compound Poisson distribution $\mathrm{CPo}(\lambda, Q)$ *is the distribution with mass function $C_Q \Pi_\lambda$:*

$$C_Q \Pi_\lambda(x) = \sum_{j=0}^{\infty} \Pi_\lambda(j) Q^{*j}(x) = \sum_{j=0}^{\infty} \frac{e^{-\lambda}\lambda^j}{j!} Q^{*j}(x), \quad x \geq 0. \tag{6}$$

In view of the Shepp-Olkin maximum entropy property (2) for the binomial distribution, a first natural conjecture might be that the compound binomial has maximum entropy among all compound Bernoulli sums $C_Q b_{\mathbf{p}}$ with a fixed mean; that is,

$$H(\mathrm{CBin}(n, \lambda/n, Q)) = \max \Big\{ H(C_Q b_{\mathbf{p}}) \ : \ \mathbf{p} \in \mathcal{P}_n(\lambda) \Big\}. \tag{7}$$

But, perhaps somewhat surprisingly, as Chi [6] has noted, (7) fails in general. For example, taking Q to be the uniform distribution on $\{1, 2\}$, $\mathbf{p} = (0.00125, 0.00875)$ and $\lambda = p_1 + p_2 = 0.01$, direct computation shows that,

$$H(\mathrm{CBin}(2, \lambda/2, Q)) < 0.090798 < 0.090804 < H(C_Q b_{\mathbf{p}}). \tag{8}$$

As the Shepp-Olkin result (2) was only seen as an intermediate step in proving the maximum entropy property of the Poisson distribution (3), we may still hope that the corresponding result remains true for compound Poisson measures, namely that,

$$H(\mathrm{CPo}(\lambda, Q)) = \sup \Big\{ H(C_Q b_{\mathbf{p}}) \ : \ \mathbf{p} \in \mathcal{P}_\infty(\lambda) \Big\}. \tag{9}$$

Again, (9) fails in general. For example, taking the same Q, λ and $\mathbf{p}$ as above, yields,

$$H(\mathrm{CPo}(\lambda, Q)) < 0.090765 < 0.090804 < H(C_Q b_{\mathbf{p}}).$$

The main purpose of the present work is to show that, despite these negative results, it is possible to provide natural, broad sufficient conditions, under which the compound binomial and compound Poisson distributions can be shown to have maximal entropy in an appropriate class of measures. Our first result, Theorem 1.4 below, states that (7) *does* hold, under certain conditions on Q and $\mathrm{CBin}(n, \lambda, Q)$:

Theorem 1.4 *If the distribution Q on $\mathbb{N}$ and the compound binomial distribution* $\mathrm{CBin}(n, \lambda/n, Q)$ *are both log-concave, then,*

$$H(\mathrm{CBin}(n, \lambda/n, Q)) = \max\Big\{H(C_Q b_{\mathbf{p}}) \ : \ \mathbf{p} \in \mathcal{P}_n(\lambda)\Big\},$$

as long as the tail of Q satisfies either one of the following properties: (a) *Q has finite support; or* (b) *Q has tails heavy enough so that, for some $\rho, \beta > 0$ and $N_0 \geq 1$, we have, $Q(x) \geq \rho^{x^\beta}$, for all $x \geq N_0$.*

The proof of the theorem is given in Section 3. As can be seen there, conditions (a) and (b) are introduced purely for technical reasons, and can probably be significantly relaxed. The notion of log-concavity, on the other hand, is central in the development of the ideas in this work. [In a different setting, log-concavity also appears as a natural condition for a different maximum entropy problem considered by Cover and Zhang [7].] Recall that the distribution P of a random variable X on $\mathbb{Z}_+$ is *log-concave* if its support is a (possibly infinite) interval of successive integers in $\mathbb{Z}_+$, and,

$$P(x)^2 \geq P(x+1)P(x-1), \quad \text{for all } x \geq 1. \tag{10}$$

We also recall that most of the commonly used distributions appearing in applications (e.g., the Poisson, binomial, geometric, negative binomial, hypergeometric logarithmic series, or Polya-Eggenberger distribution) are log-concave.

Another key property is that of ultra log-concavity; cf. [22]. The distribution P of a random variable X is *ultra log-concave* if $P(x)/\Pi_\lambda(x)$ is log-concave, that is, if,

$$xP(x)^2 \geq (x+1)P(x+1)P(x-1), \quad \text{for all } x \geq 1. \tag{11}$$

Note that the Poisson distribution as well as all Bernoulli sums are ultra log-concave.

Johnson [12] recently proved the following maximum entropy property for the Poisson distribution, generalizing (3):

$$H(\mathrm{Po}(\lambda)) = \max\Big\{H(P) \ : \ \text{ultra log-concave } P \text{ with mean } \lambda\Big\}. \tag{12}$$

Our next result (proved in Section 2) states that, as long as Q and the compound Poisson measure $\mathrm{CPo}(\lambda, Q)$ are log-concave, the same maximum entropy statement as in (12) remains valid in the compound Poisson case:

Theorem 1.5 *If the distribution Q on $\mathbb{N}$ and and the compound Poisson distribution* $\mathrm{CPo}(\lambda, Q)$ *are both log-concave, then,*

$$H(\mathrm{CPo}(\lambda, Q)) = \max\Big\{H(C_Q P) \ : \ \textit{ultra log-concave } P \textit{ with mean } \lambda\Big\}.$$

In Section 4 we give conditions under which the compound Poisson and compound Bernoulli distributions are log-concave. In particular, the results there imply the following explicit maximum entropy statements.

Example 1.6 *1. Let Q be an arbitrary log-concave distribution on $\mathbb{N}$. Then Lemma 4.1 combined with Theorem 1.4 implies that the maximum entropy property of the compound binomial distribution in equation* (7) *holds, for all λ large enough. That is, the compound binomial* $\mathrm{CBin}(n, \lambda/n, Q)$ *has maximal entropy among all compound Bernoulli sums $C_Q b_{\mathbf{p}}$ with $p_1 + p_2 + \cdots + p_n = \lambda$, as long as $\lambda \geq \frac{nQ(2)}{Q(1)^2+Q(2)}$.*

2. *Suppose Q is supported on $\{1, 2\}$, with probabilities $Q(1) = q, Q(2) = 1 - q$, and consider the class of all Bernoulli sums $b_{\mathbf{p}}$ with mean $p_1+p_2+\cdots+p_n = \lambda$. Theorem 4.2 combined with Theorem 1.5 implies that the compound Poisson maximum entropy property* (9) *holds in this case, as long as λ is large enough. In other words, the distribution* $\mathrm{CPo}(\lambda, Q)$ *has maximal entropy among all compound Bernoulli sums $C_Q b_{\mathbf{p}}$ with $p_1 + p_2 + \cdots + p_n = \lambda \geq \frac{2(1-q)}{q^2}$.*

3. *Suppose Q is geometric with parameter $\alpha \in (0, 1)$, i.e., $Q(x) = \alpha(1-\alpha)^{x-1}$ for all $x \geq 1$, and again consider the class of a Bernoulli sums $b_{\mathbf{p}}$ with mean λ. Then Theorem 4.4 combined with Theorem 1.5 implies that* (9) *holds for all large λ: The compound Poisson distribution* $\mathrm{CPo}(\lambda, Q)$ *has maximal entropy among all compound Bernoulli sums $C_Q b_{\mathbf{p}}$ with $p_1 + p_2 + \cdots + p_n = \lambda \geq \frac{2(1-\alpha)}{\alpha}$.*

Clearly, it remains an open question to give *necessary* and sufficient conditions on λ and Q for the compound Poisson and compound binomial distributions to have maximal entropy within an appropriately defined class, or even for the compound Poisson distribution to be log-concave. Section 4 ends with a conjecture, together with some supporting evidence, stating that $\mathrm{CPo}(\lambda, Q)$ is log-concave when Q is log-concave and $\lambda Q(1)^2 \geq 2Q(2)$.

2 Maximum Entropy Property of the Compound Poisson Distribution

Here we show that, if Q and the compound Poisson distribution $\mathrm{CPo}(\lambda, Q) = C_Q \Pi_\lambda$ are both log-concave, then $\mathrm{CPo}(\lambda, Q)$ has maximum entropy among all distributions of the form $C_Q P$, when P has mean λ and is ultra log-concave. Our approach is an extension of the 'semigroup' arguments of [12].

We begin by recording some basic properties of log-concave and ultra log-concave distributions:

(i) If P is ultra log-concave, then from the definitions it is immediate that P is log-concave.

(ii) If Q is log-concave, then it has finite moments of all orders; see [16, Theorem 7].

(iii) If X is a random variable with ultra log-concave distribution P, then (by (i) and (ii)) it has finite moments of all orders. Moreover, considering the covariance between the decreasing function $P(x+1)(x+1)/P(x)$ and the increasing function $x(x-1)\cdots(x-n)$, shows that the falling factorial moments of P satisfy,

$$E[(X)_n] := E[X(X-1)\cdots(X-n+1)] \leq (E(X))^n;$$

see [12] and [10] for details.

(iv) The Poisson distribution and all Bernoulli sums are ultra log-concave.

Recall the following definition from [12]:

Definition 2.1 *Given $\alpha \in [0,1]$ and a random variable $X \sim P$ on $\mathbb{Z}_+$ with mean $\lambda \geq 0$, let $U_\alpha P$ denote the distribution of the random variable,*

$$\sum_{i=1}^{X} B_i + Z_{\lambda(1-\alpha)},$$

where the B_i are i.i.d. Bern (α)*, $Z_{\lambda(1-\alpha)}$ has distribution* $\mathrm{Po}(\lambda(1-\alpha))$*, and all random variables are independent of each other and of X.*

Note that, if $X \sim P$ has mean λ, then $U_\alpha P$ has the same mean. Also, recall the following useful relation that was established in Proposition 3.6 of [12]: For all $y \geq 0$,

$$\frac{\partial}{\partial \alpha} U_\alpha P(y) = \frac{1}{\alpha} \left(\lambda(U_\alpha P(y) - U_\alpha P(y-1) - ((y+1)U_\alpha P(y+1) - y U_\alpha P(y)) \right). \tag{13}$$

Next we define another transformation of probability distributions P on $\mathbb{Z}_+$:

Definition 2.2 *Given $\alpha \in [0,1]$, a distribution P on $\mathbb{Z}_+$ and a compounding distribution Q on $\mathbb{N}$, let $U_\alpha^Q P$ denote the distribution $C_Q U_\alpha P$:*

$$U_\alpha^Q P(x) := C_Q U_\alpha P(x) = \sum_{y=0}^{\infty} U_\alpha P(y) Q^{*y}(x), \quad x \geq 0.$$

An important observation that will be at the heart of the proof of Theorem 1.5 below is that, for $\alpha = 0$, $U_0^Q P$ is simply the compound Poisson measure $\mathrm{CP}(\lambda, Q)$, while for $\alpha = 1$, $U_1^Q P = C_Q P$. The following lemma, proved in the appendix, gives a rough bound on the third moment of $U_\alpha^Q P$:

Lemma 2.3 *Suppose P is an ultra log-concave distribution with mean $\lambda > 0$ on $\mathbb{Z}_+$, and let Q be a log-concave compounding distribution on $\mathbb{N}$. For each $\alpha \in [0,1]$, let W_α, V_α be random variables with distributions $U_\alpha^Q P = C_Q U_\alpha P$ and $C_Q(U_\alpha P)^\#$, respectively, where, for any distribution R with mean ν, we write $R^\#(y) = R(y+1)(y+1)/\nu$ for its* size-biased *version. Then the third moments $E(W_\alpha^3)$ and $E(V_\alpha^3)$ are both bounded above by,*

$$\lambda q_3 + 3\lambda^2 q_1 q_2 + \lambda^3 q_1^3,$$

where q_1, q_2, q_3 denote the first, second and third moments of Q, respectively.

In [12], the characterization of the Poisson as a maximum entropy distribution was proved through the decrease of its score function. In an analogous way, following [3], we define the score function of a Q-compound random variable as follows.

Definition 2.4 *Given a distribution P on $\mathbb{Z}_+$ with mean λ, the corresponding Q-compound distribution $C_Q P$ has score function defined by:*

$$r_{1,C_Q P}(x) = \frac{\sum_{y=0}^{\infty}(y+1)P(y+1)Q^{*y}(x)}{\lambda \sum_{y=0}^{\infty} P(y) Q^{*y}(x)} - 1 = \frac{\sum_{y=0}^{\infty}(y+1)P(y+1)Q^{*y}(x)}{\lambda C_Q P(x)} - 1. \quad (14)$$

Notice that the mean of of $r_{1,C_Q P}$ with respect to $C_Q P$ is zero, and that if $P \sim \mathrm{Po}(\lambda)$ then $r_{1,C_Q P}(x) \equiv 0$. Further, when Q is the point mass at 1 this score function reduces to the "scaled score function" introduced in [18]. But, unlike the scaled score function and the alternative score function $r_{2,C_Q P}$ given in [3], this score function is not only a function of the compound distribution $C_Q P$, but also explicitly depends on P. A projection identity and other properties of $r_{1,C_Q P}$ are proved in [3].

Next we show that, if Q is log-concave and P is ultra log-concave, then the score function $r_{1,C_Q P}(x)$ is decreasing in x.

Lemma 2.5 *If P is ultra log-concave and the compounding distribution Q is log-concave, then the score function $r_{1,C_Q P}(x)$ of $C_Q P$ is decreasing in x.*

Proof First we recall Theorem 2.1 of Keilson and Sumita [17], which implies that, if Q is log-concave, then for any $m \geq n$, and for any x:

$$Q^{*m}(x+1)Q^{*n}(x) - Q^{*m}(x)Q^{*n}(x+1) \geq 0. \tag{15}$$

[This can be proved by considering Q^{*m} as the convolution of Q^{*n} and $Q^{*(m-n)}$, and writing

$$\begin{aligned} & Q^{*m}(x+1)Q^{*n}(x) - Q^{*m}(x)Q^{*n}(x+1) \\ = & \sum_l Q^{*(m-n)}(l)\Big(Q^{*n}(x+1-l)Q^{*n}(x) - Q^{*n}(x-l)Q^{*n}(x+1)\Big). \end{aligned}$$

Since Q is log-concave, then so is Q^{*n}, cf. [15], so the ratio $Q^{*n}(x+1)/Q^{*n}(x)$ is decreasing in x, and (15) follows.]

By definition, $r_{1,C_QP}(x) \geq r_{1,C_QP}(x+1)$ if and only if,

$$\begin{aligned} 0 \leq & \left(\sum_y (y+1)P(y+1)Q^{*y}(x)\right)\left(\sum_z P(z)Q^{*z}(x+1)\right) \\ & -\left(\sum_y (y+1)P(y+1)Q^{*y}(x+1)\right)\left(\sum_z P(z)Q^{*z}(x)\right) \\ = & \sum_{y,z} (y+1)P(y+1)P(z)\left[Q^{*y}(x)Q^{*z}(x+1) - Q^{*y}(x+1)Q^{*z}(x)\right]. \end{aligned} \tag{16}$$

Noting that for $y = z$ the term in square brackets in the double sum becomes zero, and swapping the values of y and z in the range $y > z$, the double sum in (16) becomes,

$$\sum_{y<z} [(y+1)P(y+1)P(z) - (z+1)P(z+1)P(y)]\left[Q^{*y}(x)Q^{*z}(x+1) - Q^{*y}(x+1)Q^{*z}(x)\right].$$

By the ultra log-concavity of P, the first square bracket is positive for $y \leq z$, and by equation (15) the second square bracket is also positive for $y \leq z$. $\square$

We remark that, under the same assumptions, and using a very similar argument, an analogous result holds for the score function r_{2,C_QP} recently introduced in [3].

Combining Lemmas 2.5 and 2.3 with equation (13) we deduce the following result, which is the main technical step in the proof of Theorem 1.5 below.

Proposition 2.6 *Let P be an ultra log-concave distribution on $\mathbb{Z}_+$ with mean $\lambda > 0$, and assume that Q and $\mathrm{CPo}(\lambda, Q)$ are both log-concave. Let W_α be a random variable with distribution $U_\alpha^Q P$, and define, for all $\alpha \in [0,1]$, the function,*

$$E(\alpha) := E[-\log C_Q\Pi_\lambda(W_\alpha)].$$

Then $E(\alpha)$ is continuous for all $\alpha \in [0,1]$, it is differentiable for $\alpha \in (0,1)$, and, moreover, $E'(\alpha) \leq 0$ for $\alpha \in (0,1)$. In particular, $E(0) \geq E(1)$.

Proof Recall that,

$$U_\alpha^Q P(x) = C_Q U_\alpha P(x) = \sum_{y=0}^{\infty} U_\alpha P(y) Q^{*y}(x) = \sum_{y=0}^{x} U_\alpha P(y) Q^{*y}(x),$$

where the last sum is restricted to the range $0 \le y \le x$, because Q is supported on $\mathbb{N}$. Therefore, since $U_\alpha P(x)$ is continuous in α [12], so is $U_\alpha^Q P(x)$, and to show that $E(\alpha)$ is continuous it suffices to show that the series,

$$E(\alpha) := E[-\log C_Q \Pi_\lambda(W_\alpha)] = -\sum_{x=0}^{\infty} U_\alpha^Q P(x) \log C_Q \Pi_\lambda(x), \tag{17}$$

converges uniformly. To that end, first observe that log-concavity of $C_Q\Pi_\lambda$ implies that $Q(1)$ is nonzero. [Otherwise, if $i > 1$ be the smallest integer i such that $Q(i) \neq 0$, then $C_Q\Pi_\lambda(i+1) = 0$, but $C_Q\Pi_\lambda(i)$ and $C_Q\Pi_\lambda(2i)$ are both strictly positive, contradicting the log-concavity of $C_Q\Pi_\lambda$.] Since $Q(1)$ is nonzero, we can bound the compound Poisson probabilities as,

$$1 \ge C_Q\Pi_\lambda(x) = \sum_y [e^{-\lambda}\lambda^y/y!] Q^{*y}(x) \ge e^{-\lambda}[\lambda^x/x!] Q(1)^x, \quad \text{for all } x \ge 1,$$

so that the summands in (17) can be bounded,

$$0 \le -\log C_Q\Pi_\lambda(x) \le \lambda + \log x! - x\log(\lambda Q(1)) \le Cx^2, \quad x \ge 1, \tag{18}$$

for a constant $C > 0$ that depends only on λ and $Q(1)$. Therefore, for any $N \ge 1$, the tail of the series (17) can be bounded,

$$0 \le -\sum_{x=N}^{\infty} U_\alpha^Q P(x) \log C_Q\Pi_\lambda(x) \le CE[W_\alpha^2 \mathbb{I}_{\{W_\alpha \ge N\}}] \le \frac{C}{N} E[W_\alpha^3],$$

and, in view of Lemma 2.3, it converges uniformly.

Therefore, $E(\alpha)$ is continuous in α, and, in particular, convergent for all $\alpha \in [0,1]$. To prove that it is differentiable at each $\alpha \in (0,1)$ we need to establish that: (i) the summands in (17) are continuously differentiable in α for each x; and (ii) the series of derivatives converges uniformly.

Since, as noted above, $U_\alpha^Q P(x)$ is defined by a finite sum, we can differentiate with respect to α under the sum, to obtain,

$$\frac{\partial}{\partial\alpha} U_\alpha^Q P(x) = \frac{\partial}{\partial\alpha} C_Q U_\alpha P(x) = \sum_{y=0}^{x} \frac{\partial}{\partial\alpha} U_\alpha P(y) Q^{*y}(x). \tag{19}$$

And since $U_\alpha P$ is continuously differentiable in $\alpha \in (0,1)$ for each x (cf. [12, Proposition 3.6] or equation (13) above), so are the summands in (17), establishing (i); in fact, they are infinitely differentiable, which can be seen by repeated applications of (13). To show that the series of derivatives converges uniformly, let α be restricted in an arbitrary open interval $(\epsilon, 1)$ for some $\epsilon > 0$. The relation (13) combined with (19) yields, for any x,

$$\begin{aligned}
&\frac{\partial}{\partial\alpha}U_\alpha^Q P(x) \\
&= \sum_{y=0}^{x}\Big(\lambda(U_\alpha P(y) - U_\alpha P(y-1) - ((y+1)U_\alpha P(y+1) - yU_\alpha P(y))\Big)Q^{*y}(x) \\
&= -\frac{1}{\alpha}\sum_{y=0}^{x}\left((y+1)U_\alpha P(y+1) - \lambda U_\alpha P(y)\right)\left(Q^{*y}(x) - Q^{*y+1}(x)\right) \\
&= -\frac{1}{\alpha}\sum_{y=0}^{x}\left((y+1)U_\alpha P(y+1) - \lambda U_\alpha P(y)\right)Q^{*y}(x) \\
&\quad + \sum_{v=0}^{x} Q(v)\frac{1}{\alpha}\sum_{y=0}^{x}\left((y+1)U_\alpha P(y+1) - \lambda U_\alpha P(y)\right)Q^{*y}(x-v) \\
&= -\frac{\lambda}{\alpha}U_\alpha^Q P(x)\left(\frac{\sum_{y=0}^{x}(y+1)U_\alpha P(y+1)Q^{*y}(x)}{\lambda U_\alpha^Q P(x)} - 1\right) \\
&\quad + \frac{\lambda}{\alpha}\sum_{v=0}^{x} Q(v)U_\alpha^Q P(x-v)\left(\frac{\sum_{y=0}^{x}(y+1)U_\alpha P(y+1)Q^{*y}(x-v)}{\lambda U_\alpha^Q P(x-v)} - 1\right) \\
&= -\frac{\lambda}{\alpha}\left(U_\alpha^Q P(x) r_{1,U_\alpha^Q P}(x) - \sum_{v=0}^{x} Q(v)U_\alpha^Q P(x-v) r_{1,U_\alpha^Q P}(x-v)\right). \qquad (20)
\end{aligned}$$

Also, for any x, by definition,

$$|U_\alpha^Q P(x) r_{1,U_\alpha^Q P}(x)| \leq C_Q(U_\alpha P)^{\#}(x) + U_\alpha^Q P(x),$$

where, for any distribution P, we write $P^{\#}(y) = P(y+1)(y+1)/\lambda$ for its size-biased version. Hence for any $N \geq 1$, equations (20) and (18) yield the bound,

$$\begin{aligned}
&\left|\sum_{x=N}^{\infty}\frac{\partial}{\partial\alpha}U_\alpha^Q P(x)\log C_Q\Pi_\lambda(x)\right| \\
&\leq \sum_{x=N}^{\infty}\frac{C\lambda x^2}{\alpha}\Big\{C_Q(U_\alpha P)^{\#}(x) + U_\alpha^Q P(x) + \sum_{v=0}^{x}Q(v)[C_Q(U_\alpha P)^{\#}(x-v) + U_\alpha^Q P(x-v)]\Big\} \\
&= \frac{2C}{\alpha}E\Big[\Big(V_\alpha^2 + W_\alpha^2 + X^2 + XV_\alpha + XW_\alpha\Big)\mathbb{I}_{\{V_\alpha\geq N,\ W_\alpha\geq N,\ X\geq N\}}\Big] \\
&\leq \frac{C'}{\alpha}\Big\{E[V_\alpha^2\mathbb{I}_{\{V_\alpha\geq N\}}] + E[W_\alpha^2\mathbb{I}_{\{W_\alpha\geq N\}}] + E[X^2\mathbb{I}_{\{X\geq N\}}]\Big\} \\
&\leq \frac{C'}{N\alpha}\Big\{E[V_\alpha^3] + E[W_\alpha^3] + E[X^3]\Big\},
\end{aligned}$$

where $C, C' > 0$ are appropriate finite constants, and the random variables $V_\alpha \sim C_Q(U_\alpha P)^{\#}$, $W_\alpha \sim U_\alpha^Q P$ and $X \sim Q$ are independent. Lemma 2.3 implies that this bound converges to zero uniformly in $\alpha \in (\epsilon, 1)$, as $N \to \infty$. Since $\epsilon > 0$ was arbitrary, this establishes that $E(\alpha)$ is differentiable for all $\alpha \in (0,1)$ and, in fact, that we can differentiate the series (17) term-by-term, to obtain,

$$
\begin{aligned}
E'(\alpha) &= -\sum_{x=0}^{\infty} \frac{\partial}{\partial \alpha} U_\alpha^Q P(x) \log C_Q \Pi_\lambda(x) \qquad (21)\\
&= \frac{\lambda}{\alpha} \sum_{x=0}^{\infty} \left(U_\alpha^Q P(x) r_{1,U_\alpha^Q P}(x) - \sum_{v=0}^{x} Q(v) U_\alpha^Q P(x-v) r_{1,U_\alpha^Q P}(x-v) \right) \log C_Q \Pi_\lambda(x)\\
&= \frac{\lambda}{\alpha} \sum_{x=0}^{\infty} U_\alpha^Q P(x) r_{1,U_\alpha^Q P}(x) \left(\log C_Q \Pi_\lambda(x) - \sum_{v=0}^{\infty} Q(v) \log C_Q \Pi_\lambda(x+v) \right),
\end{aligned}
$$

where the second equality follows from using (20) above, and the rearrangement leading to the third equality follows by interchanging the order of (second) double summation and replacing x by $x+v$.

Now we note that, exactly as in [12], the last series above is the covariance between the (zero-mean) function $r_{1,U_\alpha^Q P}(x)$ and the function $(\log C_Q \Pi_\lambda(x) - \sum_v Q(v) \log C_Q \Pi_\lambda(x+v))$, under the measure $U_\alpha^Q P$. Since P is ultra log-concave, so is $U_\alpha P$ [12], hence the score function $r_{1,U_\alpha^Q P}(x)$ is decreasing in x, by Lemma 2.5. Also, the log-concavity of $C_Q \Pi_\lambda$ implies that the second function is increasing, and Chebyshev's rearrangement lemma implies that the covariance is less than or equal to zero, proving that $E'(\alpha) \le 0$, as claimed.

Finally, the fact that $E(0) \ge E(1)$ is an immediate consequence of the continuity of $E(\alpha)$ on $[0,1]$ and the fact that $E'(\alpha) \le 0$ for all $\alpha \in (0,1)$. $\square$

Notice that, for the above proof to work, it is not necessary that $C_Q \Pi_\lambda$ be log-concave; the weaker property that $(\log C_Q \Pi_\lambda(x) - \sum_v Q(v) \log C_Q \Pi_\lambda(x+v))$ be increasing is enough.

Proof of Theorem 1.5 As in Proposition 2.6, let $W_\alpha \sim U_\alpha^Q P = C_Q U_\alpha P$, and let $D(P\|Q)$ denote the relative entropy between P and Q,

$$
D(P\|Q) := \sum_{x \ge 0} P(x) \log \frac{P(x)}{Q(x)}.
$$

Then, noting that $W_0 \sim C_Q \Pi_\lambda$ and $W_1 \sim C_Q P$, we have,

$$
\begin{aligned}
H(C_Q P) &\le H(C_Q P) + D(C_Q P \| C_Q \Pi_\lambda)\\
&= -E[\log C_Q \Pi_\lambda(W_1)]\\
&\le -E[\log C_Q \Pi_\lambda(W_0)]\\
&= H(C_Q \Pi_\lambda),
\end{aligned}
$$

where the first inequality is simply the nonnegativity of relative entropy, and the second inequality is exactly the statement that $E(1) \le E(0)$, proved in Proposition 2.6. $\square$

3 Maximum Entropy Property of the Compound Binomial Distribution

Here we prove the maximum entropy result for compound binomial random variables, Theorem 1.4. The proof, to some extent, parallels some of the arguments in [9][21][23], which rely on differentiating the compound-sum probabilities $b_{\mathbf{p}}(x)$ for a given parameter vector $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ (recall Definition 1.1 in the Introduction), with respect to an individual p_i. Using the representation,

$$C_Q b_{\mathbf{p}}(y) = \sum_{x=0}^{n} b_{\mathbf{p}}(x) Q^{*x}(y), \quad y \geq 0, \tag{22}$$

differentiating $C_Q b_{\mathbf{p}}(x)$ reduces to differentiating $b_{\mathbf{p}}(x)$, and leads to an expression equivalent to that derived earlier in (20) for the derivative of $C_Q U_\alpha P$ with respect to α.

Lemma 3.1 *Given a parameter vector $\mathbf{p} = (p_1, p_2, \ldots, p_n)$, with $n \geq 2$ and each $0 \leq p_i \leq 1$, let,*

$$\mathbf{p_t} = \left(\frac{p_1 + p_2}{2} + t, \frac{p_1 + p_2}{2} - t, p_3, \ldots, p_n \right),$$

for $t \in [-(p_1 + p_2)/2, (p_1 + p_2)/2]$. Then,

$$\frac{\partial}{\partial t} C_Q b_{\mathbf{p_t}}(x) = (-2t) \sum_{y=0}^{n} b_{\widetilde{\mathbf{p}}}(y) \left(Q^{*(y+2)}(x) - 2Q^{*(y+1)}(x) + Q^{*y}(x) \right), \tag{23}$$

where $\widetilde{\mathbf{p}} = (p_3, \ldots, p_n)$.

Proof Note that the sum of the entries of $\mathbf{p}_t$ is constant as t varies, and that $\mathbf{p_t} = \mathbf{p}$ for $t = (p_1 - p_2)/2$, while $\mathbf{p_t} = ((p_1 + p_2)/2, (p_1 + p_2)/2, p_3, \ldots, p_n)$ for $t = 0$. Writing $k = p_1 + p_2$, $b_{\mathbf{p_t}}$ can be expressed,

$$\begin{aligned} b_{\mathbf{p_t}}(y) &= \left(\frac{k^2}{4} - t^2 \right) b_{\widetilde{\mathbf{p}}}(y-2) + \left(k \left(1 - \frac{k}{2} \right) + 2t^2 \right) b_{\widetilde{\mathbf{p}}}(y-1) \\ &\quad + \left(\left(1 - \frac{k}{2} \right)^2 - t^2 \right) b_{\widetilde{\mathbf{p}}}(y), \end{aligned}$$

and its derivative with respect to t is,

$$\frac{\partial}{\partial t} b_{\mathbf{p_t}}(y) = -2t \left(b_{\widetilde{\mathbf{p}}}(y-2) - 2b_{\widetilde{\mathbf{p}}}(y-1) + b_{\widetilde{\mathbf{p}}}(y) \right).$$

The expression (22) for $C_Q b_{\mathbf{p_t}}$ shows that it is a finite linear combination of compound-sum probabilities $b_{\mathbf{p_t}}(x)$, so we can differentiate inside the sum to obtain,

$$\begin{aligned}
\frac{\partial}{\partial t} C_Q b_{\mathbf{p_t}}(x) &= \sum_{y=0}^{n} \frac{\partial}{\partial t} b_{\mathbf{p_t}}(y) Q^{*y}(x) \\
&= -2t \sum_{y=0}^{n} \left(b_{\widetilde{\mathbf{p}}}(y-2) - 2b_{\widetilde{\mathbf{p}}}(y-1) + b_{\widetilde{\mathbf{p}}}(y) \right) Q^{*y}(x) \\
&= -2t \sum_{y=0}^{n-2} b_{\widetilde{\mathbf{p}}}(y) \left(Q^{*(y+2)}(x) - 2Q^{*(y+1)}(x) + Q^{*y}(x) \right),
\end{aligned}$$

since $b_{\widetilde{\mathbf{p}}}(y) = 0$ for $y \leq -1$ and $y \geq n-1$. $\square$

Next we state and prove the equivalent of Proposition 2.6 above:

Proposition 3.2 *Suppose that the distribution Q on $\mathbb{N}$ and the compound binomial distribution* $\mathrm{CBin}(n, \lambda/n, Q)$ *are both log-concave; let $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ be a given parameter vector with $n \geq 2$, $p_1 + p_2 + \ldots + p_n = \lambda > 0$, and $p_1 \geq p_2$; let W_t be a random variable with distribution $C_Q b_{\mathbf{p_t}}$; and define, for all $t \in [0, (p_1 - p_2)/2]$, the function,*

$$E(t) := E[-\log C_Q b_{\overline{\mathbf{p}}}(W_t)],$$

where $\overline{\mathbf{p}}$ denotes the parameter vector with all entries equal to λ/n. If Q satisfies either of the conditions: (a) Q finite support; or (b) Q has tails heavy enough so that, for some $\rho, \beta > 0$ and $N_0 \geq 1$, we have, $Q(x) \geq \rho^{x^\beta}$, for all $x \geq N_0$, then $E(t)$ is continuous for all $t \in [0, (p_1 - p_2)/2]$, it is differentiable for $t \in (0, (p_1 - p_2)/2)$, and, moreover, $E'(t) \leq 0$ for $t \in (0, (p_1 - p_2)/2)$. In particular, $E(0) \geq E((p_1 - p_2)/2)$.

Proof The compound distribution $C_Q b_{\mathbf{p_t}}$ is defined by the finite sum,

$$C_Q b_{\mathbf{p_t}}(x) = \sum_{y=0}^{n} b_{\mathbf{p_t}}(y) Q^{*y}(x),$$

and is, therefore, continuous in t. First, assume that Q has finite support. Then so does $C_Q b_{\mathbf{p}}$ for any parameter vector $\mathbf{p}$, and the continuity and differentiability of $E(t)$ are trivial. In particular, the series defining $E(t)$ is a finite sum, so we can differentiate term-by-term, to obtain,

$$
\begin{aligned}
E'(t) &= -\sum_{x=0}^{\infty} \frac{\partial}{\partial t} C_Q b_{\mathbf{p_t}}(x) \log C_Q b_{\overline{\mathbf{p}}}(x) \\
&= 2t \sum_{x=0}^{\infty} \sum_{y=0}^{n-2} b_{\widetilde{\mathbf{p}}}(y) \left(Q^{*(y+2)}(x) - 2Q^{*(y+1)}(x) + Q^{*y}(x) \right) \log C_Q b_{\overline{\mathbf{p}}}(x) \qquad (24) \\
&= 2t \sum_{y=0}^{n-2} \sum_{z=0}^{\infty} b_{\widetilde{\mathbf{p}}}(y) Q^{*y}(z) \sum_{v,w} Q(v)Q(w) \Big[\log C_Q b_{\overline{\mathbf{p}}}(z+v+w) - \log C_Q b_{\overline{\mathbf{p}}}(z+v) \\
&\qquad - \log C_Q b_{\overline{\mathbf{p}}}(z+w) + \log C_Q b_{\overline{\mathbf{p}}}(z) \Big], \qquad (25)
\end{aligned}
$$

where (24) follows by Lemma 3.1. By assumption, the distribution $C_Q b_{\overline{\mathbf{p}}} = \text{CBin}(n, \lambda/n, Q)$ is log-concave, which implies that, for all z, v, w such that $z + v + w$ is in the support of $\text{CBin}(n, \lambda/n, Q)$,

$$
\frac{C_Q b_{\overline{\mathbf{p}}}(z)}{C_Q b_{\overline{\mathbf{p}}}(z+v)} \leq \frac{C_Q b_{\overline{\mathbf{p}}}(z+w)}{C_Q b_{\overline{\mathbf{p}}}(z+v+w)}.
$$

Hence the term in square brackets in equation (25) is negative, and the result follows.

Now, suppose condition (b) holds on the tails of Q. First we note that the moments of W_t are all uniformly bounded in t: Indeed, for any $\gamma > 0$,

$$
E[W_t^\gamma] = \sum_{x=0}^{\infty} C_Q b_{\mathbf{p_t}}(x) x^\gamma = \sum_{x=0}^{\infty} \sum_{y=0}^{n} b_{\mathbf{p_t}}(y) Q^{*y}(x) x^\gamma \leq \sum_{y=0}^{n} \sum_{x=0}^{\infty} Q^{*y}(x) x^\gamma \leq C_n q_\gamma, \qquad (26)
$$

where C_n is a constant depending only on n, and q_γ is the γth moment of Q, which is of course finite; recall property (ii) in the beginning of Section 2.

For the continuity of $E(t)$, it suffices to show that the series,

$$
E(t) := E[-\log C_Q b_{\overline{\mathbf{p}}}(W_t)] = -\sum_{x=0}^{\infty} C_Q b_{\mathbf{p_t}}(x) \log C_Q b_{\overline{\mathbf{p}}}(x), \qquad (27)
$$

converges uniformly. The tail assumption on Q implies that, for all $x \geq N_0$,

$$
1 \geq C_Q b_{\overline{\mathbf{p}}}(x) = \sum_{y=0}^{n} b_{\overline{\mathbf{p}}}(y) Q^{*y}(x) \geq \lambda(1-\lambda/n)^{n-1} Q(x) \geq \lambda(1-\lambda/n)^{n-1} \rho^{x^\beta},
$$

so that,

$$
0 \leq -\log C_Q b_{\overline{\mathbf{p}}}(x) \leq C x^\beta, \qquad (28)
$$

for an appropriate constant $C > 0$. Then, for $N \geq N_0$, the tail of the series (27) can be bounded,

$$
0 \leq -\sum_{x=N}^{\infty} C_Q b_{\mathbf{p_t}}(x) \log C_Q b_{\overline{\mathbf{p}}}(x) \leq C E[W_t^\beta \mathbb{I}_{\{W_t \geq N\}}] \leq \frac{C}{N} E[W_t^{\beta+1}] \leq \frac{C}{N} C_n q_{\beta+1},
$$

where the last inequality follows from (26). This obviously converges to zero, uniformly in t, therefore $E(t)$ is continuous.

For the differentiability of $E(t)$, note that the summands in (17) are continuously differentiable (by Lemma 3.1), and that the series of derivatives converges uniformly in t; to see that, for $N \geq N_0$ we apply Lemma 3.1 together with the bound (28) to get,

$$\begin{aligned}
&\left|\sum_{x=N}^{\infty} \frac{\partial}{\partial t} C_Q b_{\mathbf{p_t}}(x) \log C_Q b_{\overline{\mathbf{p}}}(x)\right| \\
&\leq \quad 2t \sum_{x=N}^{\infty} \sum_{y=0}^{n} b_{\widetilde{\mathbf{p}}}(y) \left(Q^{*(y+2)}(x) + 2Q^{*(y+1)}(x) + Q^{*y}(x)\right) C x^{\beta} \\
&\leq \quad 2Ct \sum_{y=0}^{n} \sum_{x=N}^{\infty} \left(Q^{*(y+2)}(x) + 2Q^{*(y+1)}(x) + Q^{*y}(x)\right) x^{\beta},
\end{aligned}$$

which is again easily seen to converge to zero uniformly in t as $N \to \infty$, since Q has finite moments of all orders. This establishes the differentiability of $E(t)$ and justifies the term-by-term differentiation of the series (17); the rest of the proof that $E'(t) \leq 0$ is the same as in case (a). $\square$

Note that, as with Proposition 2.6, the above proof only requires that the compound binomial distribution $\text{CBin}(n, \lambda/n, Q) = C_Q b_{\overline{\mathbf{p}}}$ satisfies a property weaker than log-concavity, namely that the function, $\log C_Q b_{\overline{\mathbf{p}}}(x) - \sum_v Q(v) \log C_Q b_{\overline{\mathbf{p}}}(x+v)$, be increasing in x.

Proof of Theorem 1.4 Assume, without loss of generality, that $n \geq 2$. If $p_1 > p_2$, then Proposition 3.2 says that, $E((p_1 - p_2)/2) \leq E(0)$, that is,

$$-\sum_{x=0}^{\infty} C_Q b_{\mathbf{p}}(x) \log C_Q b_{\overline{\mathbf{p}}}(x) \leq -\sum_{x=0}^{\infty} C_Q b_{\mathbf{p}^*}(x) \log C_Q b_{\overline{\mathbf{p}}}(x),$$

where $\mathbf{p}^* = ((p_1 + p_2)/2, (p_1 + p_2)/2, p_3, \ldots p_n)$ and $\overline{\mathbf{p}} = (\lambda/n, \ldots, \lambda/n)$. Since the expression $\sum_{x=0}^{\infty} C_Q b_{\mathbf{p_t}}(x) \log C_Q b_{\overline{\mathbf{p}}}(x)$ is invariant under permutations of the elements of the parameter vectors, we deduce that it is maximized by $\mathbf{p_t} = \overline{\mathbf{p}}$. Therefore, using, as before, the nonnegativity of the relative entropy,

$$\begin{aligned}
H(C_Q b_{\mathbf{p}}) &\leq H(C_Q b_{\mathbf{p}}) + D(C_Q b_{\mathbf{p}} \| C_Q b_{\overline{\mathbf{p}}}) \\
&= -\sum_{x=0}^{\infty} C_Q b_{\mathbf{p}}(x) \log C_Q b_{\overline{\mathbf{p}}}(x) \\
&\leq -\sum_{x=0}^{\infty} C_Q b_{\overline{\mathbf{p}}}(x) \log C_Q b_{\overline{\mathbf{p}}}(x) \\
&= H(C_Q b_{\overline{\mathbf{p}}}) \;=\; H(\text{CBin}(n, \lambda/n, Q)),
\end{aligned}$$

as claimed. $\square$

4 Conditions for Log-Concavity

Theorems 1.5 and 1.4 state that log-concavity is a sufficient condition for compound binomial and compound Poisson distributions to have maximal entropy within a natural class. Here we give examples of when log-concavity holds; if the results in this section can be strengthened (in particular, if Conjecture 4.5 can be proved), then the class of maximum entropy distributions will be accordingly widened.

Below we show that a compound Bernoulli sum is log-concave if the parameters are sufficiently large, and that compound Bernoulli sums and compound Poisson distributions are log-concave if Q is either supported only on the set $\{1, 2\}$ or is geometric.

Lemma 4.1 *Suppose Q is a log-concave distribution on $\mathbb{N}$.*

(i) *The compound Bernoulli distribution* $\mathrm{CBern}\,(p, Q)$ *is log-concave if and only if* $p \geq \frac{1}{1+Q(1)^2/Q(2)}$.

(ii) *The compound Bernoulli sum distribution $C_Q b_{\mathbf{p}}$ is log-concave as along as all the elements p_i of the parameter vector $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ satisfy $p_i \geq \frac{1}{1+Q(1)^2/Q(2)}$.*

Proof Let Y have distribution $\mathrm{CBern}\,(p, Q)$. Since Q is log-concave itself, the log-concavity of $\mathrm{CBern}\,(p, Q)$ is equivalent to the inequality, $\Pr(Y = 1)^2 \geq \Pr(Y = 2)\Pr(Y = 0)$, which states that, $(pQ(1))^2 \geq (1 - p)pQ(2)$, and this is exactly the assumption of (i).

The assertion in (ii) follows from (i), since the sum of independent log-concave random variables is log-concave; see, e.g., [15]. $\square$

Next we examine conditions under which a compound Poisson measure is log-concave. Our argument is based, in part, on the some of the ideas in Johnson and Goldschmidt [13], and also in Wang and Yeh [26], where transformations that preserve log-concavity are studied.

Note that, unlike for the Poisson distribution, it is not the case that every compound Poisson distribution $\mathrm{CPo}(\lambda, Q)$ is log-concave. Indeed, for any distribution P, considering the difference, $C_Q P(1)^2 - C_Q P(0) C_Q P(2)$, shows that a necessary condition for $C_Q P$ to be log-concave is that,

$$(P(1)^2 - P(0)P(2))/P(0)P(1) \geq Q(2)/Q(1)^2. \tag{29}$$

Taking P to be the $\mathrm{Po}(\lambda)$ distribution, a necessary condition for $\mathrm{CPo}(\lambda, Q)$ to be log-concave is that,

$$\lambda \geq \frac{2Q(2)}{Q(1)^2}, \tag{30}$$

while for $P = b_{\mathbf{p}}$, a necessary condition for the compound Bernoulli sum $C_Q b_{\mathbf{p}}$ to be log-concave is,

$$\sum_i \frac{p_i}{1-p_i} + \left(\sum_i \frac{p_i^2}{(1-p_i)^2}\right)\left(\sum_i \frac{p_i}{1-p_i}\right)^{-1} \geq \frac{2Q(2)}{Q(1)^2},$$

which, by Jensen's inequality, will hold as long as, $\sum_i p_i \geq 2Q(2)/Q(1)^2$.

Theorem 4.2 *Let Q be a distribution supported on the set $\{1,2\}$.*

(i) *The compound Poisson distribution* $\mathrm{CPo}(\lambda, Q)$ *is log-concave for all* $\lambda \geq \frac{2Q(2)}{Q(1)^2}$.

(ii) *The distribution $C_Q P$ is log-concave for any ultra log-concave distribution P with support on $\{0, 1, \ldots, N\}$ (where N may be infinite), which satisfies, $(x+1)P(x+1)/P(x) \geq 2Q(2)/Q(1)^2$ for all $x = 0, 1, \ldots, N$.*

Note that, the second condition in (ii) is equivalent to requiring that $NP(N)/P(N-1) \geq 2Q(2)/Q(1)^2$ if N is finite, or that $\lim_{x\to\infty}(x+1)P(x+1)/P(x) \geq 2Q(2)/Q(1)^2$ if N is infinite.

Proof Writing $R(y) = y!P(y)$, we know that $C_Q P(x) = \sum_{y=0}^{x} R(y) \left(Q^{*y}(x)/y!\right)$. Hence, the log-concavity of $C_Q P(x)$ is equivalent to showing that,

$$\sum_r \frac{Q^{*r}(2x)}{r!} \sum_{y+z=r} R(y)R(z)\binom{r}{y}\left(\frac{Q^{*y}(x)Q^{*z}(x)}{Q^{*r}(2x)} - \frac{Q^{*y}(x+1)Q^{*z}(x-1)}{Q^{*r}(2x)}\right) \geq 0, \tag{31}$$

for all $x \geq 2$, since the case of $x = 1$ was dealt with previously by equation (29). In particular, for (i), taking $P = \mathrm{Po}(\lambda)$, it suffices to show that for all r and x, the function,

$$g_{r,x}(k) := \sum_{y+z=r} \binom{r}{y} \frac{Q^{*y}(k)Q^{*z}(2x-k)}{Q^{*r}(2x)}$$

is unimodal as a function of k (since $g_{r,x}(k)$ is symmetric about x).

In the general case (ii), writing $Q(2) = p = 1 - Q(1)$, we have, $Q^{*y}(x) = \binom{y}{x-y} p^{x-y}(1-p)^{2y-x}$, so that,

$$\binom{r}{y} \frac{Q^{*y}(k)Q^{*z}(2x-k)}{Q^{*r}(2x)} = \binom{2x-r}{k-y}\binom{2r-2x}{2y-k}, \tag{32}$$

for any p. Now, following [13, Lemma 2.4] and [26, Lemma 2.1], we use summation by parts to show that the inner sum in (31) is positive for each r (except for $r = x$ when x is odd), by case-splitting according to the parity of r.

(a) For $r = 2t$, we rewrite the inner sum of equation (31) as,

$$\sum_{s=0}^{t} (R(t+s)R(t-s) - R(t+s+1)R(t-s-1)) \times$$
$$\left(\sum_{y=t-s}^{t+s} \left(\binom{2x-r}{x-y}\binom{2r-2x}{2y-x} - \binom{2x-r}{x+1-y}\binom{2r-2x}{2y-x-1}\right)\right),$$

where the first term in the above product is positive by the ultra log-concavity of P (and hence log-concavity of R), and the second term is positive by Lemma 4.3 below.

(b) Similarly, for $x \neq r = 2t+1$, we rewrite the inner sum of equation (31) as,

$$\sum_{s=0}^{t} (R(t+s+1)R(t-s) - R(t+s+2)R(t-s-1)) \times$$
$$\left(\sum_{y=t-s}^{t+1+s} \left(\binom{2x-r}{x-y}\binom{2r-2x}{2y-x} - \binom{2x-r}{x+1-y}\binom{2r-2x}{2y-x-1} \right) \right),$$

where the first term in the product is positive by the ultra log-concavity of P (and hence log-concavity of R) and the second term is positive by Lemma 4.3 below.

(c) Finally, in the case of $x = r = 2t+1$, substituting $k = x$ and $k = x+1$ in (32), combining the resulting expression with (31), and noting that $\binom{2r-2x}{u}$ is 1 if and only if $u = 0$ (and is zero, otherwise), we see that the inner sum becomes, $-R(t+1)R(t)\binom{2t+1}{t}$, and the summands in (31) reduce to,

$$-\frac{p^x R(t)R(t+1)}{(t+1)!t!}.$$

However, the next term in the outer sum of equation (31), $r = x+1$, gives

$$\begin{aligned} &\frac{p^{x-1}(1-p)^2}{2(2t)!}\left[R(t+1)^2\left(2\binom{2t}{t} - \binom{2t}{t+1} \right) - R(t)R(t+2)\binom{2t}{t} \right] \\ \geq\ & \frac{p^{x-1}(1-p)^2}{2(2t)!} R(t+1)^2 \left(\binom{2t}{t} - \binom{2t}{t+1} \right) = \frac{p^{x-1}(1-p)^2}{2(t+1)!t!} R(t+1)^2. \end{aligned}$$

Hence, the sum of the first two terms is positive (and hence the whole sum is positive) if $R(t+1)(1-p)^2/(2p) \geq R(t)$.

If P is Poisson(λ), this simply reduces to equation (30), otherwise we use the fact that $R(x+1)/R(x)$ is decreasing. $\square$

Lemma 4.3 (a) *If $r = 2t$, for any $0 \leq s \leq t$, the sum,*

$$\sum_{y=t-s}^{t+s} \left(\binom{2x-r}{x-y}\binom{2r-2x}{2y-x} - \binom{2x-r}{x+1-y}\binom{2r-2x}{2y-x-1} \right) \geq 0.$$

(b) *If $x \neq r = 2t+1$, for any $0 \leq s \leq t$, the sum,*

$$\sum_{y=t-s}^{t+1+s} \left(\binom{2x-r}{x-y}\binom{2r-2x}{2y-x} - \binom{2x-r}{x+1-y}\binom{2r-2x}{2y-x-1} \right) \geq 0.$$

Proof The proof is in two stages; first we show that the sum is positive for $s = t$, then we show that there exists some S such that, as s increases, the increments are positive for $s \leq S$ and negative for $s > S$. The result then follows, as in [13] or [26].

For both (a) and (b), note that for $s = t$, equation (32) implies that the sum is the difference between the coefficients of T^x and T^{x+1} in $f_{r,x}(T) = (1+T^2)^{2x-r}(1+T)^{2r-2x}$. Since $f_{r,x}(T)$ has degree $2x$ and has coefficients which are symmetric about T^x, it is enough to show that the coefficients form a unimodal sequence. Now, $(1+T^2)^{2x-r}(1+T)$ has coefficients which do form a unimodal sequence. Statement S_1 of Keilson and Gerber [16] states that any binomial distribution is strongly unimodal, which means that it preserves unimodality on convolution. This means that $(1+T^2)^{2x-r}(1+T)^{2r-2x}$ is unimodal if $r - x \geq 1$, and we need only check the case $r = x$, when $f_{r,x}(T) = (1+T^2)^r$. Note that if $r = 2t$ is even, the difference between the coefficients of T^x and T^{x+1} is $\binom{2t}{t}$, which is positive.

In part (a), the increments are equal to $\binom{2x-2t}{x-t+s}\binom{4t-2x}{2t-2s-x}$ multiplied by the expression,

$$2 - \frac{(x-t-s)(2t-2s-x)}{(x+1-t+s)(2t+2s-x+1)} - \frac{(x-t+s)(2t+2s-x)}{(x+1-t-s)(2t-2s-x+1)},$$

which is positive for s small and negative for s large, since placing the term in brackets over a common denominator, the numerator is of the form $(a - bs^2)$.

Similarly, in part (b), the increments equal $\binom{2x-2t-1}{x-t+s}\binom{4t+2-2x}{2t-2s-x}$ times the expression,

$$2 - \frac{(x-t-s-1)(2t-2s-x)}{(x+1-t+s)(2t+2s-x+3)} - \frac{(x-t+s)(2t+2+2s-x)}{(x-t-s)(2t+1-2s-x)},$$

which is again positive for s small and negative for s large. □

Theorem 4.4 *Let Q be a geometric distribution on $\mathbb{N}$. Then $C_Q P$ is log-concave for any distribution P which is log-concave and satisfies the condition* (29).

Proof If Q is geometric with mean $1/\alpha$, then, $Q^{*y}(x) = \alpha^y(1-\alpha)^{x-y}\binom{x-1}{y-1}$, which implies that,

$$C_Q P(x) = \sum_{y=0}^{x} P(y)\alpha^y(1-\alpha)^{x-y}\binom{x-1}{y-1}.$$

Condition (29) ensures that $C_Q P(1)^2 - C_Q P(0) C_Q P(2) \geq 0$, so, taking $z = y - 1$, we need only prove that the sequence,

$$C(x) := C_Q P(x+1)/(1-\alpha)^x = \sum_{z=0}^{x} P(z+1)\left(\frac{\alpha}{1-\alpha}\right)^{z+1}\binom{x}{z}$$

is log-concave. However, this follows immediately from [15, Theorem 7.3], which proves that if $\{a_i\}$ is a log-concave sequence, then so is $\{b_i\}$, defined by $b_i = \sum_{j=0}^{i}\binom{i}{j}a_j$. □

Finally, based on the discussion in the beginning of this section, the above results, and some calculations of the quantities, $C_Q\Pi_\lambda(x)^2 - C_Q\Pi_\lambda(x-1)C_Q\Pi_\lambda(x+1)$ for small x, we make the following conjecture:

Conjecture 4.5 *The compound Poisson measure* $\mathrm{CPo}(\lambda, Q)$ *is log-concave, as long as* Q *is log-concave and* $\lambda Q(1)^2 \geq 2Q(2)$.

The condition $\lambda Q(1)^2 \geq 2Q(2)$ is, of course, necessary; recall the argument leading to equation (30) above.

In closing, we list some known results that are related to this conjecture and may be useful in proving (or disproving) it:

1. Theorem 2.3 of Steutel and van Harn [24] shows that, if $\{iQ(i)\}$ is a decreasing sequence, then $\mathrm{CPo}(\lambda, Q)$ is a unimodal distribution (recall that log-concavity implies unimodality). Interestingly, the same condition provides a dichotomy of results in compound Poisson approximation bounds as developed in [2]: If $\{iQ(i)\}$ is decreasing the bounds are of the same form and order as in the simple Poisson case, while if it is not the bounds are much larger.

2. Theorem 3.2 of Cai and Willmot [5] shows that if $\{Q(i)\}$ is decreasing then the distribution function of the compound Poisson distribution $\mathrm{CPo}(\lambda, Q)$ is log-concave.

3. A conjecture similar to Conjecture 4.5 is that, for log-concave Q, if $\mathrm{CPo}(\lambda, Q)$ is log-concave, then so is $\mathrm{CPo}(\mu, Q)$, for all $\mu \geq \lambda$. Theorem 4.9 of Keilson and Sumita [17] proves the related result that, if Q is log-concave, then, for any n, the ratio,
$$\frac{C_Q \Pi_\lambda(n)}{C_Q \Pi_\lambda(n+1)} \quad \text{is decreasing in } \lambda.$$

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Appendix

Proof of Lemma 2.3 Recall that, as stated in properties (*ii*) and (*iii*) in the beginning of Section 2, Q has finite moments of all orders, and that the nth falling factorial moment of any ultra log-concave random variable Y with distribution R on $\mathbb{Z}_+$ is bounded above by $(E(Y))^n$. Now for an arbitrary ultra log-concave distribution R, define random variables $Y \sim R$ and $Z \sim C_Q R$. If r_1, r_2, r_3 denote the first three moments of $Y \sim R$, then,

$$\begin{aligned} E(Z^3) &= q_3 r_1 + 3q_1 q_2 E[(Y)_2] + q_1^3 E[(Y)_3] \\ &\leq q_3 r_1 + 3q_1 q_2 r_1^2 + q_1^3 r_1^3. \end{aligned} \tag{33}$$

Since the map U_α preserves ultra log-concavity [12], if P is ultra log-concave then so is $R = U_\alpha P$, so that (33) gives the required bound for the third moment of W_α, upon noting that the mean of the distribution $U_\alpha P$ is equal to λ.

Similarly, size-biasing preserves ultra log-concavity; that is, if R is ultra log-concave, then so is $R^\#$, since $R^\#(x+1)(x+1)/R^\#(x) = (R(x+2)(x+2)(x+1))/(R(x+1)(x+1)) = R(x+2)(x+2)/R(x+1)$ is also decreasing. Hence, $R' = (U_\alpha P)^\#$ is ultra log-concave, and (33) applies in this case as well. In particular, noting that the mean of $Y' \sim R' = (U_\alpha P)^\# = R^\#$ can be bounded in terms of the mean of $Y \sim R$ as,

$$E(Y') = \sum_x x \frac{(x+1)U_\alpha P(x+1)}{\lambda} = \frac{E[(Y)_2]}{E(Y)} \leq \frac{\lambda^2}{\lambda} = \lambda,$$

the bound (33) yields the required bound for the third moment of V_α. □

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