Unfolding the Entropy Power Inequality

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***Abstract*—We insert an interesting quantity involving rearrangements in between the two sides of the entropy power inequality, thereby refining it.**

I. Main Result

The entropy power inequality (EPI) is a basic and powerful tool in information theory, and also has relevance to probability theory and mathematical physics. Applications include converse parts of various coding theorems, the information-theoretic understanding of probabilistic limit theorems, and functional analytic inequalities like the Gaussian logarithmic Sobolev inequality. Numerous variants and generalizations of the EPI have been proved, discussed, and applied to various problems in the literature. However, to our knowledge, there are no known direct refinements of the EPI in the sense that a natural information-theoretic quantity is inserted between the left and right sides of the classical formulations of the inequality. Our main contribution is to provide such a refinement.

The refinement is stated in terms of the notion of the spherically symmetric decreasing rearrangement of a random vector (or its probability density function), which we now define. For a Borel set A with volume $\mathrm{Vol}(A)$, define its spherically symmetric rearrangement A^* as the open ball with center at the origin, and volume exactly equal to $\mathrm{Vol}(A)$. Here we use the convention that if $|A| = 0$, then $A^* = \emptyset$. Now for a measurable non-negative function f, define its spherically symmetric decreasing rearrangement f^* by:

$$f^*(y) = \int_0^{+\infty} \{y \in B_t^*\} dt$$

where $B_t = \{x : f(x) > t\}$. A basic property of spherically symmetric decreasing rearrangement is that L^p norms are preserved [1]:

$$\|f\|_p = \|f^*\|_p$$

for $1 \leq p < +\infty$. In particular, if f is a probability density, so is f^*.

Theorem 1. *Let $f_i, i = 1, 2, \cdots, n$ be n probability densities on $\mathbb{R}^d$ and $f_i^*, i = 1, 2, \cdots, n$ be the spherically symmetric decreasing rearrangements of the corresponding densities. Then*

$$h(f_1 \star f_2 \star \cdots \star f_n) \geq h(f_1^* \star f_2^* \star \cdots \star f_n^*),$$

as long as the two entropies exist.

If we write X_i^* for a random vector drawn from the density f_i^*, and assume that all random vectors are drawn independently of each other, Theorem 1 says in more customary information-theoretic notation that

$$h(X_1 + \ldots + X_n) \geq h(X_1^* + \ldots + X_n^*).$$

II. Why the Main Result strengthens the EPI

In order to make the comparison, we state the following standard version of the entropy power inequality [2]. (Although we focus on real-valued random variables in the discussion below for simplicity, it can be extended easily to the case of $\mathbb{R}^d$-valued random vectors.)

Theorem 2. *Let X_1 and X_2 be two independent $\mathbb{R}$-valued random variables with finite differential entropies, and finite variance. Let Z_1 and Z_2 be two independent Gaussians such that*

$$h(X_i) = h(Z_i), \quad i = 1, 2.$$

Then

$$h(X_1 + X_2) \geq h(Z_1 + Z_2).$$

We also need the following lemmata, which appear to be new, although the proofs are not difficult.

Lemma 3. *If one of $h(X)$ and $h(X^*)$ is well defined, then so is the other one and we have*

$$h(X) = h(X^*).$$

Lemma 4. *For any real random variable X,*

$$\mathit{Var}(X^*) \leq \mathit{Var}(X).$$

First note that from Lemma 3, it follows that

$$h(X_i^*) = h(X_i) = h(Z_i), \quad i = 1, 2.$$

Furthermore, Lemma 4 implies that X_1^* and X_2^* have finite variance, and therefore by the usual EPI (i.e., Theorem 2), we have that

$$h(X_1^* + X_2^*) \geq h(Z_1 + Z_2). \tag{1}$$

On the other hand, Theorem 1 gives

$$h(X_1 + X_2) \geq h(X_1^* + X_2^*). \tag{2}$$

From (2) and (1), we see that we have inserted the quantity $h(X_1^* + X_2^*)$ between the two sides of the EPI as stated in Theorem 2, as asserted.

III. Remarks

The details of the proof of Theorem 1 (which is independent of the EPI or any Fisher information, MMSE or entropy differentiation arguments, and relies on rearrangement inequalities developed in [3]), as well as various applications, will be contained in a forthcoming paper by the authors. Theorem 1 can also be generalized; in particular, it turns out that it has a natural extension to Rényi entropies of all orders.

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