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Chapter 3

The moment generating function method

MGF::MGF

SECTION 3.1 *introduces the MGF (moment generating function) method for bounding tail probabilities.*

SECTION 3.3 *illustrates the MGF method for the simplest case, the normal distribution. The normal is the prototype for the subgaussian distributions, which will be discussed in Chapter 7.*

*SECTION 3.4 *ponders the question, What do we lose if we use the subgaussian tail bound for the normal in place of better bounds that are found in the literature?*

SECTION 3.5 *derives tail bounds for the Poisson distribution. The omnipresent convex function* $\psi_{\text{Benn}}()$ *puts in an appearance. The Poisson is the prototype for the Bennett inequalities, which will be derived in Chapter 8.*

SECTION 3.6 *establishes tail bounds for the gamma distribution, the prototype for the Bernstein inequalities, which will be derived in Chapter 8.*

SECTION 3.7 *establishes very good bounds for the tails of the Binomial distribution, which look a lot like a fancier version of the bounds for the Poisson. These bounds also work for Poisson-Binomial distributions and other sums of independent random variables taking values in* $[0, 1]$*. Both results follow via Jensen's inequality from the convexity of the exponential function.*

SECTION 3.8 *shows that the tail bounds derived by the MGF method for the hypergeometric distribution (sampling without replacement) are smaller than the bounds for the corresponding Binomial (sampling with replacement).*

3.1 Tail bounds from the moment generating function

MGF::S:method

Much modern statistical theory relies on a handful of probabilistic inequalities, often in the form of bounds on tail probabilites or concentration inequalities. This Chapter introduces one of the main methods for establishing such bounds. By way of illustration, the method is applied to derive bounds for several well studied cases, which provide the prototypes for a handful of very useful tail bounds.

The method uses the MGF, $M_X(\lambda) := \mathbb{P}e^{\lambda X} = e^{L_X(\lambda)}$, to get upper bounds for $\mathbb{P}\{X \geq x\}$. Remember from Section 2.3 that L_X is infinitely differentiable and convex on the set $\{\lambda \in \mathbb{R} : M_X(\lambda) < \infty\}$.

From the fact that the exp() function is everywhere nonnegative and $\exp(\lambda(X - x)) \geq 1$ when $X \geq x$ and $\lambda \geq 0$ we have

\E@ mgf.upper.tail <1>
$$\mathbb{P}\{X \geq x\} \leq \inf_{\lambda \geq 0} \mathbb{P}e^{\lambda(X-x)} = \inf_{\lambda \geq 0} e^{-\lambda x} M_X(\lambda) = \exp\left(\inf_{\lambda \geq 0} L_X(\lambda) - \lambda x\right).$$

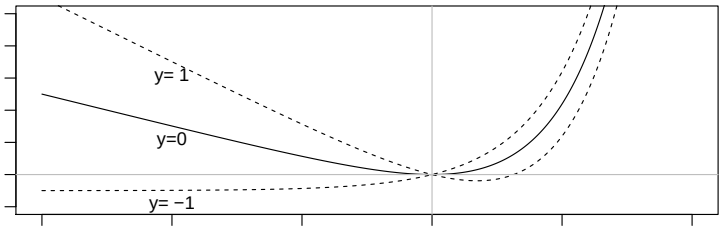
Similarly, $\exp(\lambda(X + x)) \geq 1$ if $X \leq -x$ and $\lambda \leq 0$, so that

\E@ mgf.lower.tail <2>
$$\mathbb{P}\{X \leq -x\} \leq \inf_{\lambda \leq 0} \mathbb{P}e^{\lambda(X+x)} = \exp\left(\inf_{\lambda \leq 0} L_X(\lambda) + \lambda x\right).$$

Remark. This inequality can also be derived from <1> applied to bound $\mathbb{P}\{-X \geq -x\}$ by $\inf_{\lambda \geq 0} e^{\lambda(-x)} M_{-X}(\lambda) = \inf_{\lambda \leq 0} e^{-\lambda x} M_X(\lambda)$.

The analysis is simplified if we assume that X has been centered to have $\mathbb{P}X = 0$ and L_X is finite in a neighborhood of the origin, for then $L'_X(0) = \mathbb{P}X = 0$ and the convex function L_X is minimized at the origin. Equivalently, we can just replace X by $X - \mathbb{P}X$.

$$\lambda \mapsto L_{X-\mathbb{P}X}(\lambda) - \lambda y$$



Remark. The picture actually shows the case where $X \sim \text{Poisson}(1)$.

The multiplication of $M_{X-\mathbb{P}X}$ by $e^{-\lambda y}$ tilts the convex function $L_{X-\mathbb{P}X}$ by $-\lambda y$, which ensures that $L_{X-\mathbb{P}X}(\lambda) - \lambda y$ achieves its global minimium on the half line $\{\lambda \in \mathbb{R} : \lambda \geq 0\}$ if $y > 0$ and on the half line $\{\lambda \in \mathbb{R} : \lambda \leq 0\}$ if $y < 0$. For the purposes of <1> and <2> we no longer have to consciously think about the sign of y; the infimum in both cases reduces to minimization over the whole real line and everything can be done by brute force Calculus.

In short, for $x > 0$ we have

\E@ LX.upper <3>
$$\mathbb{P}\{X - \mathbb{P}X \geq x\} \leq \exp\left(-\Lambda(x)\right),$$

\E@ LX.lower <4>
$$\mathbb{P}\{X - \mathbb{P}X \leq -x\} \leq \exp\left(-\Lambda(-x)\right),$$

where

\E@ Lam.def <5>
$$\begin{aligned}\Lambda(y) &:= -\inf_{\lambda\in\mathbb{R}}\left(L_{X-\mathbb{P}X}(\lambda) - \lambda y\right) = -\inf_{\lambda\in\mathbb{R}}\left(L_X(\lambda) - \lambda(y + \mathbb{P}X)\right)\\ &= \sup_{\lambda\in\mathbb{R}}\left(\lambda y - L_{X-\mathbb{P}X}(\lambda)\right) = \sup_{\lambda\in\mathbb{R}}\left(\lambda(y + \mathbb{P}X) - L_X(\lambda)\right),\end{aligned}$$

for all $y \in \mathbb{R}$. Effectively, this means we need only search for the global solution to $L'_X(\lambda) = y + \mathbb{P}X$ to determine Λ, except in those pesky cases where the infimum of the convex function $\lambda \mapsto L_X(\lambda) - \lambda(y + \mathbb{P}X)$ is approached as λ tends to $\pm\infty$.

Remark. The nonnegativity of Λ comes from the zero contribution at $\lambda = 0$. The second expression for $\Lambda(y)$ identifies it as as $L^*_{X-\mathbb{P}X}(y)$, the Fenchel-Legendre conjugate of the convex function $L_{X-\mathbb{P}X}$. Exciting as the recognition of this conjugate in a probability bound might be, it does not seem to help much in the actual calculation for a given X. Everything comes down to an exercise in Calculus and convexity, which can be worked through without any knowledge of the material in Section 2.4.

*3.2 Behavior of the tail bound near the origin

MGF::S:local

Even though the tail bounds are not particularly useful for small t it is illuminating to see how the moments of $X - \mathbb{P}X$ affect Λ when its MGF is finite in a neighborhood of the origin.

The MGF gets its name from the coefficients in its power series expansion

$$M_{X-\mathbb{P}X}(\lambda) = \mathbb{P}e^{\lambda(X-\mathbb{P}X)} = 1 + \sum\nolimits_{k\in\mathbb{N}} \mu_k \lambda^k / k! \qquad \text{where } \mu_k = \mathbb{P}\left(X - \mathbb{P}X\right)^k.$$

The quantities μ_k is often called the ***kth central moment*** to distinguish it from $\mathbb{P}X^k$. Note that $\mu_1 = 0$ and $\mu_2 = \text{var}(X)$. The quantity $\mu_3/\mu_2^{3/2} = \mathbb{P}(X - \mathbb{P}X)^3/\text{var}(X)^{3/2}$ is called the ***skewness*** of the distribution.

The function $L_{X-\mathbb{P}X} = \log M_{X-\mathbb{P}X}$ also has a power series expansion,

$$L_{X-\mathbb{P}X}(\lambda) = \sum\nolimits_{k\in\mathbb{N}} \kappa_k \lambda^k/k! \qquad .$$

The coefficients κ_k are called the (central?) ***cumulants*** and $L_{X-\mathbb{P}X}$, not surprisingly, is the ***cumulant generating function***. Note that there is no κ_0, because $L_{X-\mathbb{P}X}(0) = \log M_{X-\mathbb{P}X}(0) = 0$.

The cumulants can be related to the moments by equating coefficients in power series expansions. Here is how it works for the first three cumulants.

$$\begin{aligned}
&1 + \mu_2\lambda^2/2! + \mu_3\lambda^3/3! + O(\lambda^4) = \exp\left(\sum\nolimits_{k\in\mathbb{N}} \kappa_k\lambda^k/k!\right) \\
&= 1 + \left(\kappa_1\lambda + \kappa_2\lambda^2/2! + \kappa_3\lambda^3/3! + O(\lambda^4)\right) \\
&\quad + \left(\kappa_1\lambda + \kappa_2\lambda^2/2! + \kappa_3\lambda^3/3! + O(\lambda^4)\right)^2 \\
&\quad + \left(\kappa_1\lambda + \kappa_2\lambda^2/2! + \kappa_3\lambda^3/3! + O(\lambda^4)\right)^3 + O(\lambda^4) \\
&= 1 + \kappa_1\lambda + \kappa_2\lambda^2/2! + \kappa_3\lambda^3/3! + \left(\kappa_1^2\lambda^2 + \kappa_1\kappa_2\lambda^3\right) + \left(\kappa_1^3\lambda^3\right) + O(\lambda^4) \\
&= 1 + \lambda\left(\kappa_1\right) + \frac{\lambda^2}{2!}\left(\kappa_2 + 2\kappa_1^2\right) + \frac{\lambda^3}{3!}\left(\kappa_3 + 6\kappa_1\kappa_2\right) + O(\lambda^4).
\end{aligned}$$

It follows that $\kappa_1 = 0$ and $\kappa_2 = \mu_2$ and $\kappa_3 = \mu_3$.

Remark. Don't get too excited and leap to the conclusion that cumulants are the same as moments. If I hadn't centered the distribution to zero expected value then κ_1 would not be zero and μ_k would be a nasty-looking polynomial in $\kappa_1, \ldots, \kappa_k$. By repeated substitutions we could then write κ_k as another nasty-looking polynomial in the non-central moments $\mathbb{P}X, \ldots, \mathbb{P}X^k$. Even with the centering the remaining cumulants get messier: $\kappa_4 = X^2 - (\mathbb{P}X^2)^2$ and the expression for κ_{10} is a sum of 12 terms.

Back to tail probabilities. Remember that $\Lambda(x)$ is usually obtained by maximizing $\lambda x - L_{X-\mathbb{P}X}(\lambda)$ with respect to λ, with the task coming down to solving

$$t = L'_{X-\mathbb{P}X}(\lambda) = \kappa_2\lambda + \kappa_3\lambda^2/2! + \kappa_4\lambda^3/3! + \ldots \qquad .$$

This regularity suggests that the maximizing value λ_t be expressible as a power series $\sum_{k\in\mathbb{N}} a_k t^k/k!$. The a_k coefficients can be determined by another

exercise in coefficient matching. First note that

$$\lambda_t = a_1 t + a_2 \frac{t^2}{2!} + a_3 \frac{t^3}{3!} + O(t^4), \quad \lambda_t^2 = a_1^2 t^2 + a_1 a_2 t^3 + O(t^4), \quad \lambda_t^3 = a_1^3 t^3 + O(t^4).$$

Thus

$$\begin{aligned} t &= \kappa_2 \left(a_1 t + a_2 \frac{t^2}{2!} + a_3 \frac{t^3}{3!} \right) + \frac{\kappa_3}{2!} \left(a_1^2 t^2 + a_1 a_2 t^3 \right) + \frac{\kappa_4}{3!} (a_1^3 t^3) + O(t^4) \\ &= t\,(\kappa_2 a_1) + \frac{t^2}{2!} \left(\kappa_2 a_2 + \kappa_3 a_1^2 \right) + \frac{t^3}{3!} \left(\kappa_2 a_3 + 3\kappa_3 a_1 a_2 + \kappa_4 a_1^3 \right) + O(t^4), \end{aligned}$$

implying

$$a_1 = 1/\kappa_2, \qquad a_2 = -\kappa_3/\kappa_2^3, \qquad a_3 = \text{something}.$$

It now follows that

$$\begin{aligned} \Lambda(t) &= t \left(a_1 t + a_2 t^2/2 \right) - \frac{\kappa_2}{2!} \left(a_1^2 t^2 + a_1 a_2 t^3 \right) - \frac{\kappa_3}{3!} \left(a_1^3 t^3 \right) + O(t^4) \\ &= \frac{t^2}{2\kappa_2} - \frac{\kappa_3 t^3}{6\kappa_2^3} + O(t^4) \qquad \text{near the origin.} \end{aligned}$$

The contributions from a_3 and κ_4 get absorbed into the $O(t^4)$.

Remark. You might be wondering why I bothered expanding λ_t as a cubic once I had realized the a_3 term would be absorbed into the $O(t^4)$. As a wise friend once advised me, it is always a good idea to expand an approximating series out to at least one term more than one thinks is necessary. A classic example of a failure to expand far enough can be found in the famous paper of Pearson (1900, page 165), which led to many years of argument between him and R. A. Fisher over the number of degrees of freedom for a χ^2 goodness of fit test. See Cochran (1952) for a very clear discussion of Pearson's error.

MGF::Binomial.local <6> **Example.** Suppose $X \sim \text{BIN}(n, p)$. Then

$$\mathbb{P}X = np, \quad \kappa_2 = \mathbb{P}(X - np)^2 = npq, \quad \kappa_3 = \mathbb{P}(X - np)^3 = npq(q - p)$$

so that

$$\Lambda(t) = \frac{t^2}{2npq} - \frac{t^3(q-p)}{6(npq)^2} + O(t^4) \qquad \text{near the origin.}$$

This approximation shows that $\Lambda(t) \geq t^2/(2npq)$ if $p \geq 1/2$ and $0 \leq t \approx 0$, in which case

$$\mathbb{P}\{X \geq np + t\} \leq e^{-\Lambda(t)} \leq \exp\left(-\frac{t^2}{2npq} \right) \qquad 0 \leq t \approx 0.$$

In fact, as will be shown in Section 3.7, the inequality holds for all $t \geq 0$ if $p \geq 1/2$. The local property implied by negative skewness suggests a subgaussian upper tail; the convexity of the ψ_{Benn} function will transform the local suggestion into a global inequality. □

3.3 Normal

MGF::S:normal

The MGF method is cleanest for the normal distribution. As this Section shows, the method leads to bounds comparable to very sharp inequalities that can be derived using special properties of the normal.

MGF::normal <7> **Example.** If X has a $N(\mu, \sigma^2)$ distribution then $M(\lambda) = \exp(\lambda\mu + \sigma^2\lambda^2/2)$ is finite for all real λ. For $x \geq 0$ inequality <1> gives

$$\begin{aligned}\mathbb{P}\{X \geq \mu + \sigma x\} &\leq \inf_{\lambda \geq 0} \exp(-\lambda(\mu + \sigma x) + \lambda\mu + \lambda^2\sigma^2/2) \\ &= \exp(-x^2/2) \qquad \text{for all } x \geq 0,\end{aligned}$$

the minimum being achieved by $\lambda = x/\sigma$. Analogous arguments, with $X - \mu$ replaced by $\mu - X$, give an analogous bound for the lower tail,

$$\mathbb{P}\{X \leq \mu - \sigma x\} \leq \exp(-x^2/2) \qquad \text{for all } x \geq 0,$$

leading to the inequality $\mathbb{P}\{|X - \mu| \geq \sigma x\} \leq 2e^{-x^2/2}$, which shows that the distribution of X is concentrated near μ.

Remark. Of course the algebra would have been a tad simpler if I had worked with the standardized variable $(X - \mu)/\sigma$. I did things the messier way in order make the point that if Y is any random variable, not necessarily normally distributed, for which

$$M_Y(\lambda) = \mathbb{P}e^{\lambda Y} \leq e^{\nu\lambda + \lambda^2\tau^2/2} \qquad \text{for all } \lambda \geq 0$$

then

\E@ subg.upper.tail <8> $$\mathbb{P}\{Y \geq \nu + \tau x\} \leq e^{-x^2/2} \qquad \text{for } x \geq 0.$$

Here ν and τ need not equal $\mathbb{P}Y$ and $\sqrt{\text{var}(Y)}$; instead I will refer to them as the location and scale parameters. As you will see in Chapter 7, under mild assumptions the one-sided bound on M_Y implies only that the $\mathbb{P}Y \leq \nu$ and $\mathbb{P}(Y - \nu)^2 \leq \tau^2$, with $\mathbb{P}Y = \nu$ if the bound on the MGF holds for all $\lambda \in \mathbb{R}$.

Inequality <8> is sometimes called a ***subgaussian bound for the upper tail***. However, the term 'subgaussian' is also often used in a looser sense as an indication that a tail probability decreases at

an $\exp(-Ct^2)$ rate for some positive constant C. See, for example, Section 3.6 for the upper tail of $\sqrt{X}$ when X has a GAMMA(α) distribution, although the centering constant $\sqrt{\alpha}$ is larger than $\mathbb{P}\sqrt{X}$. Even sloppier is the habit (which I have) of referring to a tail bound as 'approximately subgaussian' in particular regions. See, for example, the discussion of the upper tail for the GAMMA(α) distribution. The sloppy justification for such a habit is that some applications for tail bounds involve only deviations in the 'approximately subgaussian' region.

For an X distributed $N(0,1)$, Example <7> gives

\E@ normal.tail1 <9> $$\bar{\Phi}(x) := \mathbb{P}\{X \geq x\} \leq B(x) := e^{-x^2/2} \qquad \text{for } x \geq 0.$$

*3.4 How sharp is the MGF bound for the $N(0,1)$?

MGF::S:sharp.normal

How good is the upper bound $B(x)$, derived in the previous Section, and what do we lose if we replace $\bar{\Phi}$ by its upper bound B? The message from the next two examples is: not bad and not much, especially if we are mostly interested in large values of x. If you are the trusting type, you can safely skip this Section.

MGF::conf.int <10> **Example.** Anyone who has taken an introductory Statistics course knows that if $T \sim N(\theta, 1)$ under a $\mathbb{P}_\theta$ model then, to two decimal places accuracy,

$$\mathbb{P}_\theta\{T - 1.96 \leq \theta \leq T + 1.96\} = 0.95.$$

That is, the range $T \pm 1.96$ is a 95% confidence interval for θ; the interval $T \pm 1.96$ will contain θ with 95% probability.

Remark. Those of you familiar with the tricky interpretation of a confidence interval, will know why, if the model is correct, it is ok to say that the interval contains θ with probability 0.95 as a general statement about the behavior of the random variable T but it is not ok to say the same thing after a particular value of T is observed.

Consider the effect of using the upper bound $B(x)$ instead of $\bar{\Phi}(x)$ when constructing a confidence interval. It is certainly true under the $N(\theta, 1)$ model that

$$\mathbb{P}_\theta\{T \pm c \text{ fails to contain } \theta\} \leq 2B(c) = 2\exp(-c^2/2).$$

You could think of this bound as a conservative confidence assertion: with probability at worst $2B(c)$ the interval $T \pm c$ fails to contain θ. As the following table (with values rounded to two decimal places) shows, $2B(1.96) = 2 \times 14.65\%$. As judged by B, the range $T \pm 1.96$ will contain θ with probability at least 70.7%, which is not very comforting given that the nominal value is 95%. For the 90% interval, the conservative value, $1 - 2 \times 25.85\% = 48.3\%$, is even worse. It takes a great stretching of the imagination to describe either conclusion as 'not bad'.

x	1.64	1.96	2.45	2.58	2.72	3.26
$\bar{\Phi}(x)$	5%	2.5%	0.72%	0.5%	0.33%	0.06 %
$\exp(-x^2/2)$	25.85%	14.65%	5%	3.62%	2.5%	0.5 %

There is another way to use the upper bound. Instead of stretching imagination we could stretch the interval, from $T \pm 1.96$ to $T \pm c$ with $c = 2.72$. For this interval, the B bound assures coverage of at least 95%. Looking on the bright side, I think the increase from 1.96 to 2.72 is not too high a price to pay for an appreciable relaxation of the modeling assumptions from normal to subgaussian.

The compromise would look even better when the failure probability is smaller. For example, under the strict $N(\theta, 1)$ assumption $T \pm 2.58$ is a 99% confidence interval for θ and $T \pm 3.26$ has, according to B, probability at least 99% of containing θ. □

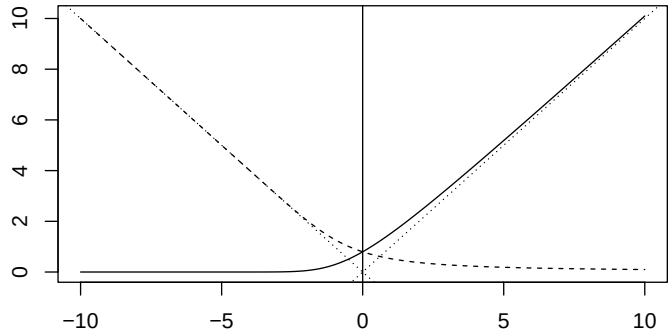
Sharper tail bounds than $\bar{\Phi}(x) \le \exp(-x^2/2)$ are possible if we exploit further properties of the normal distribution, properties that are not shared by all subgaussian distributions. In fact there is a literature going back over two centuries that contains numerous facts about $\bar{\Phi}$, including several upper and lower bounds. These bounds often depend on the identity $d\phi(x)/dx = -x\phi(x)$, where $\phi(x) = \exp(-x^2/2)/\sqrt{2\pi}$ is the $N(0,1)$. Here are the basic facts. See Problems [2] for proofs.

(i) $\bar{\Phi}(x) = \phi(x)/\rho(x)$ with ρ a convex, strictly increasing function on $\mathbb{R}$ with $\rho(x) \to 0$ as $x \to -\infty$ and $\rho(x)/x \to 1$ as $x \to \infty$. The function $\log(\rho(\cdot))$ is concave.

(ii) The function $r(x) = \rho(x) - x$ is positive, convex, strictly decreasing on $\mathbb{R}^+$ with $r(0) = \sqrt{2/\pi} \approx 0.798$ and $r(x) \to 0$ as $x \to \infty$.

(iii) The function $\mathcal{R}(x) = \bar{\Phi}(x)/\phi(x) = 1/\rho(x)$ is often called the Mills ratio, for not particularly compelling reasons. See the Notes.

The functions ρ (solid line), r (dashed line), and $\pm x$ (dotted line)



For large x the function $\bar{\Phi}(x)$ behaves like $\phi(x)/x$. More precisely,

\E@ Laplace.normal.approx <11>

$$\left(1-x^{-2}\right)\frac{\phi(x)}{x} < \bar{\Phi}(x) < \frac{\phi(x)}{x} \qquad \text{for all } x>0,$$

a result essentially due to Laplace. Equivalently, $x < \rho(x)$ for all $x > 0$ (actually, for all x because ρ is positive) and $\rho(x) < x/(1-x^{-2})$ for $x > 1$. The inequalities can be derived by integrating $\int_x^\infty$ through the pointwise bounds (for $t > 0$)

$$\begin{aligned} -\frac{d}{dt}\left[(t^{-1}-t^{-3})\phi(t)\right] &= (1-3t^{-4})\phi(t) \\ &< \phi(t) \\ &< (1+t^{-2})\phi(t) = -\frac{d}{dt}\left[t^{-1}\phi(t)\right]. \end{aligned}$$

The inequalities in <11> are quite useless for x near 0: in the limit as $x \to 0$ they deliver the unsurprising fact that $-\infty < 1/2 < +\infty$. Problem [1] gives the more general version of Laplace's approximation, obtaining upper and lower bounds for $\bar{\Phi}(x)$ that are of the form $p(1/x)\phi(x)$, with $p(\cdot)$ a polynomial. Again these bounds are informative only for large x.

For $x > 1$ inequality <11> can be rewritten as

\E@ normal.tail2 <12>

$$\begin{aligned} \log\bar{\Phi}(x) &= -x^2/2 - \log\left(x\sqrt{2\pi}\right) - \eta(x) \\ &\text{where } 0 \le \eta(x) \le -\log(1-x^{-2}) \le 2x^{-2} \text{ for } x \ge \sqrt{2}. \end{aligned}$$

The bound $B(x)$ has captured the $-x^2/2$, which is much more important than the $\log\left(x\sqrt{2\pi}\right) + \eta(x)$ when x becomes large.

The literature also contains many sharper bounds. For example,

$$(3x+\sqrt{x^2+8})/4 < \rho(x) \le \left(x+\sqrt{x^2+4}\right)/2 \qquad \text{for all } x \in \mathbb{R},$$

the lower bound coming from Birnbaum (1942) and the upper bound from Sampford (1953). See Problems [6] and [5] for proofs.

Remark. None of the bounds is accurate enough for moderate x to serve as a basis for numerical calculation of normal tail probabilities. These days, the bounds are just of theoretical interest.

The next Example illustrates what we can gain by working with <12> instead of <9> when considering a maximum of normals.

MGF::max.normal <13> **Example.** Suppose $Z_1, \ldots, Z_n$ are random variables, each distributed $N(0,1)$ but, for the moment, not necessarily independent. Define $M_n = \max_{i \le n} Z_i$.

A union bound gives some control for the tail:

$$\mathbb{P}\{M_n > x\} \le \sum\nolimits_i \mathbb{P}\{Z_i > x\} = n\bar{\Phi}(x) \le n \exp(-x^2/2) \qquad \text{for } x > 0.$$

In particular, $\mathbb{P}\{M_n > \sqrt{2\log(n) + 2c}\,\} \le e^{-c}$ for each $c \ge 0$. Roughly speaking, with high probability M_n should be not much bigger than $a_n := \sqrt{2\log n}$. Of course we get no companion lower bound for M_n from a one sided inequality <9>.

The crude union bound has the advantage of being unaffected by possible dependence between the Z_i's. It also has the disadvantage that the upper bound can be excessively large. For example, in the extreme case where $Z_i = Z_1$ for all i any bound that involves n would be superfluous.

Remark. The union bound $\mathbb{P}\left(\cup_i A_i\right) \le \sum_i \mathbb{P}A_i$ is quite good if the events A_i are independent and $\sum_i \mathbb{P}A_i$ is small. See Problem [10]. It is part of the folklore that if the A_i's are 'almost' independent then the union bound is 'almost' quite good, with the meaning of 'almost' being problem specific.

If the Z_i's are actually independent the union bound can be replaced by an equality:

$$\begin{aligned}
\mathbb{P}\{M_n \le x\} &= \prod\nolimits_i \mathbb{P}\{Z_i \le x\} = \left(1 - \bar{\Phi}(x)\right)^n \\
&= \exp\left(n \log(1 - \bar{\Phi}(x))\right) \\
&= \exp\left(-n\bar{\Phi}(x) - n \sum\nolimits_{j \ge 2} \left(\bar{\Phi}(x)\right)^j / j\right) \\
&= \exp\left(-n\bar{\Phi}(x) - R_n(x)\right) \qquad \text{where } 0 \le R_n(x) \le \frac{n\bar{\Phi}(x)^2}{2(1 - \bar{\Phi}(x))}.
\end{aligned}$$

\E@ indepN(0,1) <14>

If we choose x_n so that $n\bar{\Phi}(x_n) \to \infty$ then we get $\mathbb{P}\{M_n > x_n\} \to 1$ as $n \to \infty$. To guide the choice of x_n, use approximation <12>:

$$\log\left(n\bar{\Phi}(x_n)\right) = \log n - \tfrac{1}{2}x_n^2 - \log(x_n\sqrt{2\pi}) - \eta(x_n).$$

For x_n of the form $a_n - w/a_n$ we have

\E@ approx.nPhibar <15>
$$n\bar{\Phi}(x_n) = \exp\left(w - \log\left(a_n\sqrt{2\pi}\right) - \epsilon_n\right)$$

where $\epsilon_n = -\log\left(1 - w/a_n^2\right) - \eta\left(a_n - w/a_n\right)$ is close to 0 if $|w|/a_n$ is small. For example, if $w = 2\log(a_n) \approx \log\log n$ then $n\bar{\Phi}(x_n) \approx \log n$ so that $\mathbb{P}\{M_n \le x_n\}$ is of order n^{-1} or smaller. With probability close to 1, the maximum M_n is greater than $a_n - (\log\log n)/a_n$. Similarly, the median of M_n is very slightly larger and $M_n \le a_n$ with probability close to 1. □

Remark. Inequalities <14> and <15> are the basis for the classical fact that $a_n(M_n - b_n)$, with $b_n = a_n - \left(\log\log n + \log(4\pi)\right)/(2a_n)$, converges in distribution. See Leadbetter, Lindgren, and Rootzén (1983, Theorem 1.5.3).

3.5 Poisson

MGF::S:Poisson

The normal distribution represents the prototype for the class of subgaussian distributions. In a similar way the Poisson provides the prototype for a class of distributions that might be (but are not) called subPoisson. These distributions behave like subgaussians for moderately large deviations from the mean but decrease only a little faster than the exponential further out in the tails. The Bennett inequalities in Chapter 8 will provide further examples.

Recall that a random variable Y has a POISSON(θ) distribution if

$$\mathbb{P}\{Y = k\} = e^{-\theta}\theta^k/k! \qquad \text{for } k = 0, 1, \ldots.$$

The parameter θ must be strictly positive. The random variable $X = Y - \theta$ has a zero expected value with $\operatorname{var}(X) = \theta$ and

$$L_X(\lambda) = \theta(e^\lambda - 1 - \lambda) = \theta\mathbb{f}(\lambda) \qquad \text{for all } \lambda \in \mathbb{R}.$$

As explained in Section 3.1, we can derive both upper and lower tail bounds from the function

$$-\Lambda(y) = \inf_{\lambda\in\mathbb{R}}\left(L_X(\lambda) - y\lambda\right) = \theta\inf_{\lambda\in\mathbb{R}}\left(\mathbb{f}(\lambda) - \lambda y/\theta\right).$$

Notice the appearance of our friend $\mathbb{f}$ from Section 2.2. Its comrade $\mathbb{h}$ is coming soon.

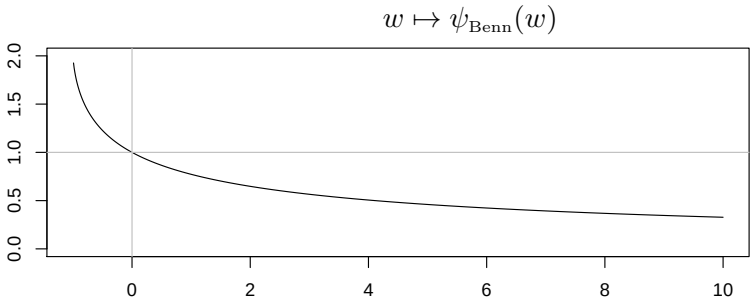
Temporarily write w for y/θ. Note that $\mathbb{f}(\lambda)-\lambda w = e^\lambda - 1 - \lambda(1+w)$ has derivative $e^\lambda - (1+w)$, which is zero at $\lambda = \log(1+w)$ if $w > -1$. If $w = -1$ the derivative is everywhere strictly positive, so that the infimum of -1 is approached as $\lambda \to -\infty$. If $w < -1$ then $\mathbb{f}(\lambda) - \lambda w = e^\lambda - 1 - \lambda(1+w)$, which approaches $-\infty$ as $\lambda \to -\infty$. In summary, $\inf_{\lambda\in\mathbb{R}} (\mathbb{f}(\lambda) - \lambda w)$ equals

\E@ Poisson.min <16>

$$\begin{cases} -(1+w)\log(1+w) + w & \text{if } w > -1\text{; achieved at } \lambda = \log(1+w) \\ -1 & \text{if } w = -1\text{; approached as } \lambda \to \infty \\ -\infty & \text{if } w < -1\text{; approached as } \lambda \to \infty \end{cases}$$

Remark. If you have read Section 2.4 you will realize that I am here repeating the calculation that showed $\mathbb{h}$ is the Fenchel-Legendre conjugate of $\mathbb{f}$.

If you have read Section 2.2 you will also know that $\mathbb{h}(w) = \frac{1}{2}w^2\psi_{\text{Benn}}(w)$ for $w \geq -1$, where $\psi_{\text{Benn}}(\cdot)$ is a convex, decreasing function on $[-1,\infty)$ with $\psi_{\text{Benn}}(0) = 1$. For large w the value of $\psi_{\text{Benn}}(w)$ decreases like $2w^{-1}\log(w)$.



Thus

$$\Lambda(y) = \theta\mathbb{h}(y/\theta) = \begin{cases} \dfrac{y^2}{2\theta}\psi_{\text{Benn}}(y/\theta) & \text{if } y \geq -\theta \\ \infty & \text{if } y < -\theta \end{cases},$$

which translates into

$$\begin{aligned} \mathbb{P}\{X \geq x\} &\leq \exp\left(-\frac{x^2}{2\theta}\psi_{\text{Benn}}(x/\theta)\right) \qquad \text{for } x \geq 0 \\ \mathbb{P}\{X \leq -x\} &\leq \exp\left(-\frac{x^2}{2\theta}\psi_{\text{Benn}}(-x/\theta)\right) \qquad \text{for } 0 \leq x \leq \theta \\ \mathbb{P}\{X \leq -x\} &\leq 0 \qquad \text{for } x > \theta. \end{aligned}$$

The third inequality is reassuring because $\mathbb{P}\{X < -\theta\} = 0$. The first inequality shows that the upper tail decreases like a subgaussian in the range $0 \le x \ll \theta$, because $\psi_{\text{Benn}}(x/\theta) \approx 1$ for x/θ near 0, but that the tail decay becomes more like $\exp(-x\log(x/\theta))$ further out into the tail. The inequality for the lower tail is more interesting, because $\psi_{\text{Benn}}(w) > 1$ for $-1 \le w < 0$. The lower tails drop off even faster than one might expect from the $N(\theta,\theta)$ approximation to the POISSON(θ). This can be interpreted as a skewness effect: $\mathbb{P}X^3$ is the coefficient of $\lambda^3/3!$ in the power series expansion of

$$\begin{aligned}\mathbb{P}e^{\lambda X} &= \exp\Big(\theta(e^\lambda - 1 - \lambda)\Big) \\ &= 1 + \theta\left(\frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \dots\right) + \frac{\theta^2}{2!}\left(\frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \dots\right)^2 + \dots,\end{aligned}$$

which is positive. The distribution of $X - \theta$ puts more mass to the right of the origin than the $N(0,\theta)$. That fact slows down the decay in the upper tails but improves the rate of decay in the lower tail.

The MGF tail bound for the Poisson does not quite capture the actual behavior of the probabilities. The deficiency parallels what happens with the normal, where $\bar\Phi(x)$ decreases like $e^{-x^2}/\left(\sqrt{2\pi}x\right)$ for large x but the MGF method captures only the $\exp(-x^2/2)$. I hope you drew the conclusion from Section 3.3 that the failure was not fatal.

For $X = Y - \theta$ with $Y \sim$ POISSON(θ), the $\exp(-\theta\hbar(y/\theta))$ tail bound compares favorably with the probability calculated by means of Stirling's formula (see Section 2.5) for $k \in \mathbb{N}$:

$$k! = \sqrt{2\pi}k^{k+1/2}e^{-k+r_k} \qquad \text{where } \frac{1}{12k+1} < r_k < \frac{1}{12k}.$$

If $k = \theta + y$ then

$$\begin{aligned}\log\Big(\sqrt{2\pi k}\,\mathbb{P}\{Y = k\}\Big) &= -\theta + k\log(\theta) - k\log(k) + k - r_k \\ &= y - (\theta+y)\log(1 + y/\theta) - r_k \\ &= -\theta\hbar(y/\theta) - r_k.\end{aligned}$$

Once again the MGF method has successfully captured the most important term, $-\theta\hbar(y/\theta)$, in the exponent.

3.6 Gamma and chi-squared

MGF::S:Gamma

Suppose $X = Y - \alpha$ where Y has a GAMMA(α) distribution, that is, the distribution on $\mathbb{R}^+$ that has density $f_\alpha(x) = x^{\alpha-1}e^{-x}/\Gamma(\alpha)$ with respect

to Lebesgue measure. The positive parameter α is often called the *shape parameter*. The expected value and variance of Y both equal α and

$$\mathbb{P}e^{\lambda Y} = \int_0^\infty \frac{x^{\alpha-1}e^{-x(1-\lambda)}}{\Gamma(\alpha)}\,dx = (1-\lambda)^{-\alpha} \qquad \text{for } \lambda < 1.$$

Thus $L_X(\lambda) = -\alpha\lambda - \alpha\log(1-\lambda)$ for $\lambda < 1$, so that

\E@ upper.gamma.tail <17> $$\mathbb{P}\{X \le t\} \le \exp\Bigl(-\Lambda(t)\Bigr) \qquad \text{for } t \ge 0$$

\E@ lower.gamma.tail <18> $$\mathbb{P}\{X \le -t\} \le \exp\Bigl(-\Lambda(-t)\Bigr) \qquad \text{for } \alpha > t \ge 0$$

where

$$\begin{aligned}\Lambda(y) &= \sup_{0\le\lambda<1} \lambda(\alpha+y) + \alpha\log(1-\lambda) \\ &= \begin{cases} y - \alpha\log(1+y/\alpha) & \text{if } \alpha+y>0\text{; achieved at } \lambda = y/(\alpha+y) \\ \infty & \text{if } \alpha+y \le 0\text{; approached as } \lambda\to-\infty \end{cases}\end{aligned}$$

For $t \ge 0$ this gives

$$\log\mathbb{P}\{Y \ge \alpha+t\} = \log\mathbb{P}\{X \ge t\} \le -\Lambda(t) \approx \begin{cases} -t^2/(2\alpha) & \text{if } t \text{ is near } 0 \\ -t & \text{if } t \text{ is large} \end{cases}.$$

and for $0 \le t < \alpha$ it gives

$$\log\mathbb{P}\{Y \le \alpha - t\} = \log\mathbb{P}\{X \le -t\} \le -\Lambda(-t) \le -t^2/(2\alpha).$$

The lower tail is actually subgaussian.

Boucheron, Lugosi, and Massart (2013, page 28) pointed out that the tails can also be bounded by first using an upper bound for the logMGF of X when $|\lambda| < 1$:

\E@ gamma.BML <19> $$\alpha^{-1}\log\mathbb{P}e^{\lambda X} = -\log(1-\lambda) - \lambda = \sum\nolimits_{i\ge2}\frac{\lambda^i}{i} \le \sum\nolimits_{i\ge2}\frac{|\lambda|^i}{2} = \frac{\lambda^2}{2(1-|\lambda|)}.$$

They referred to the one-sided analog of this inequality for $0 < \lambda < 1$ as a $\Gamma_+(\alpha,1)$ bound, which controls the upper tail, and for $-1 < \lambda < 0$ as a $\Gamma_-(\alpha,1)$ bound, which controls the lower tail. They also introduced a second parameter, for scaling. I'll focus mostly on the upper tail. When I need to contemplate negative λ, as in Example <24>, I'll work directly from <19>.

MGF::subGamma.def <20> **Definition.** *For constants $\alpha > 0$ and $\beta > 0$, interpret $W \in \text{subGamma}(\alpha, \beta)$ to mean*

$$\mathbb{P}e^{\lambda W} \le \exp\left(\frac{\alpha\lambda^2/2}{1-\beta\lambda}\right) \qquad \text{for } 0 \le \beta\lambda < 1.$$

□ *Abbreviate $\text{subGamma}(\alpha, 1)$ to $\text{subGamma}(\alpha)$.*

Remarks.

(i) The presence of the $|\lambda|$ on the right-hand side of <19> for $Y \sim \text{Gamma}(\alpha)$ implies $\pm(Y - \alpha) \in \text{subGamma}(\alpha)$. As the lower tail is actually subgaussian, the $\pm$ is a bit misleading.

(ii) For Y with a $\text{Gamma}(\alpha)$ distribution we have $Y - \mathbb{P}Y \in \text{subGamma}(\alpha)$, which suggests that subGamma might require a zero expected value. However, just as you saw with an attempt to define a one-sided subgaussian property, the best we can infer is a nonnegative expected value: if $W \in \text{subGamma}(\alpha)$ then

$$M_W(\lambda) = 1 + \lambda\mathbb{P}W + o(\lambda) \le \exp\left(\alpha\lambda^2 + o(\lambda^3)\right) = 1 + O(\lambda^2)$$

for λ near 0, which implies $\mathbb{P}W \le 0$.

Notice that if $W \in \text{subGamma}(\alpha, \beta)$ then $W/\beta \in \text{subGamma}(\alpha/\beta^2)$. Equivalently, if $Y \in \text{subGamma}(\alpha)$ then $\beta Y \in \text{subGamma}(\alpha\beta^2, \beta)$. I find it cleaner to derive general theory with $\beta = 1$ and then deduce the corresponding $\text{subGamma}(\alpha, \beta)$ facts by rescaling. For example, if $Y \sim \chi^2_k$ then $Y/2 \sim \text{Gamma}(k/2)$, so that $Y/2 - k/2 \in \text{subGamma}(k/2)$, which implies $Y - k \in \text{subGamma}(2k, 2)$.

MGF::subGamma.tail <21> **Theorem.** *If $W \in \text{subGamma}(\alpha)$ and $t \ge 0$ then*

$$\mathbb{P}\{W \ge t\} \le e^{-H_1(t,\alpha)} \le e^{-H_2(t,\alpha)}$$

where

$$H_1(t,\alpha) = \left(t + \alpha - \sqrt{2t\alpha + \alpha^2}\right) = \frac{-t^2}{(t+\alpha) + \sqrt{2t\alpha + \alpha^2}} \quad ,$$

$$H_2(t,\alpha) = \frac{-t^2}{2(t+\alpha)} \quad .$$

Proof.

$$\begin{aligned}\log \mathbb{P}\{W \geq t\} &= \inf_{0\leq\lambda<1}\left(-t\lambda + \frac{\alpha\lambda^2}{2(1-\lambda)}\right)\\ &= \inf_{0<s\leq 1}\left(-t(1-s) + \alpha\frac{1-2s+s^2}{2s}\right)\\ &= \inf_{0<s\leq 1}\left((t+\alpha/2)s - (t+\alpha) + \alpha/(2s)\right)\end{aligned}$$

The replacement of λ by $1-s$ makes it easier to calculate the derivative, $t+\alpha/2-\alpha/(2s^2)$, which is zero when $s=\sqrt{\alpha/(2t+\alpha)}$. That value gives the first expression for H_1. The final inequality comes from $\alpha^2+2\alpha t \leq (\alpha+t)^2$. □

For future reference, if $W \in \text{subGamma}(\alpha,\beta)$ and $t \geq 0$ then

\E@ H1.subGamma <22>
$$\mathbb{P}\{W \geq t\} \leq e^{-H_1(t/\beta,\alpha/\beta^2)} = \exp\left(-\frac{t^2}{\alpha+\beta t+\sqrt{\alpha^2+2t\alpha\beta}}\right)$$

\E@ H2.subGamma <23>
$$\leq e^{-H_2(t/\beta,\alpha/\beta^2)} = \exp\left(-\frac{t^2}{2(\alpha+\beta t)}\right).$$

MGF::weighted.chi2 <24>
Example. (Laurent and Massart, 2000, Lemma 1) Consider the weighted sum $W=\sum_{j=1}^k a_j(Z_j^2-1)$ where $a=(a_1,\ldots,a_k)\in\mathbb{R}^k$ and $Z=(Z_1,\ldots,Z_k)$ has a $N(0,I_k)$ distribution. As usual, define

$$|a|_\infty := \max_j |a_j| \quad \text{AND} \quad |a|_2 := \sqrt{\sum\nolimits_j a_j^2}.$$

Each Z_j^2 has a χ_1^2 distribution and $\frac{1}{2}Z_j^2 \sim \text{Gamma}(1/2)$. From inequality <19> we have

$$\mathbb{P}e^{\lambda W} \leq \prod_{j=1}^k \exp\left(\frac{\lambda^2 a_j^2}{1-|2\lambda a_j|}\right) \leq \exp\left(\frac{\lambda^2 2|a|_2^2/2}{1-2|\lambda|\,|a|_\infty}\right) \qquad \text{for } |\lambda| < \frac{1}{2|a|_\infty}.$$

Consequently $W \in \text{subGamma}(2|a|_2^2, 2|a|_\infty)$ and, for example,

$$\mathbb{P}\{W \geq t\} \leq \exp\left(\frac{-t^2}{4|a|_2^2+4|a|_\infty t}\right) \qquad \text{for } t \geq 0.$$

The quantity on the right-hand side of the last inequality is unchanged if we replace each a_j by $-a_j$. Thus $\mathbb{P}\{-W \geq t\}$ is bounded above by the same quantity and $\mathbb{P}\{|W| \geq t\}$ is bounded above by twice that quantity. □

To compare the three tail bounds from <17> and <21> it helps to isolate the effect of α by writing $\Lambda(t) = \alpha R_0(t/\alpha)$ and $H_j(t,\alpha) = \alpha R_j(t/\alpha)$ for $j = 1, 2$, for $x \geq 0$. Then

$$\begin{aligned} 0 \leq R_0(t) &:= t - \log(1+t) \\ &\leq R_1(t) := 1 + t - \sqrt{1+2t} = \frac{t^2}{1+t+\sqrt{1+2t}} \\ &\leq R_2(t) := t^2/(2t+2). \end{aligned}$$

The inequalities are all strict for $t > 0$. All three functions R_0, R_1, and R_2 have first derivatives that are positive and increasing; all are convex and strictly increasing. Near the origin $R_0(t) = t^2/-t^3/3+0(t^3)$ and both $R_1(t)$ and $R_2(t)$ behave like $t^2/2-t^3/2+o(t^3)$. As $t \to \infty$ both $R_0(t)/t$ and $R_1(t)/t$ converge to 1 but $R_2(t) \to 1/2$. If one is not too worried about the constants in the exponent there is not much difference between the three tail bounds.

It might appear that there is little point in recording the H_1 tail bound when it differs so little from the H_2 tail bound. However H_1 does give a more pleasing result if we rearrange the bound by solving $R_1(t) = w$ for a fixed $w > 0$. There is a unique positive $t = w + \sqrt{2w}$, the larger of the two roots of the quadratic $t \mapsto (1+t-w)^2 = 1+2t$. In particular, $\alpha R_1(t/\alpha) = w$ if $t = w + \sqrt{2\alpha w}$. With such a change of variable, the H_1 form of inequality from Theorem <21> takes the neat form (Boucheron et al., 2013, page 29)

\E@ BLM29 <25> $$\mathbb{P}\{W \geq w + \sqrt{2\alpha w}\} \leq e^{-w} \qquad \text{for } w \geq 0 \text{ if } W \in \text{subGamma}(\alpha).$$

Remark. You should carry out the analogous calculation for R_2. The result is not as elegant or useful.

In particular, if $Y \sim \text{Gamma}(\alpha)$ then

\E@ root.gamma <26>
$$\begin{aligned} \mathbb{P}\{\sqrt{Y} \geq \sqrt{\alpha} + \sqrt{w}\} &= \mathbb{P}\{Y \geq \alpha + 2\sqrt{\alpha w} + w\} \\ &\leq \mathbb{P}\{Y - \alpha \geq w + \sqrt{2\alpha w}\} \leq e^{-w} \qquad \text{for } w \geq 0. \end{aligned}$$

Substituting t for $\sqrt{w}$ we get $\mathbb{P}\{\sqrt{Y} \geq \sqrt{\alpha} + t\} \leq e^{-t^2}$, an example of a "subgaussian bound" for an upper tail beyond a point strictly larger than the mean: Jensen's inequality gives $\mathbb{P}\sqrt{Y} < \sqrt{\alpha}$.

MGF::chi2 <27> **Example.** If $W \sim \chi^2_k$ then $W/2 \sim \text{Gamma}(k/2)$ and inequality <26> implies

$$\mathbb{P}\{\sqrt{W} \geq \sqrt{k} + t\} \leq e^{-t^2/2} \qquad \text{for } t \geq 0.$$

In particular, $\mathbb{P}\{\sqrt{W} \geq 2\sqrt{k}\,\} \leq e^{-k/2}$ is a convenient bound when precise constant don't matter and k is large.

As an application, suppose $Z_1, \ldots, Z_n$ are independent $N(0, I_d)$ random vectors. Then we have $\mathbb{P}\,\|Z_i\| \leq \sqrt{\mathbb{P}\,\|Z_i\|^2} = \sqrt{d}$ and $\sum_{i\leq n}\|Z_i\|^2 \sim \chi^2_{nd}$. From the inequality $n^{-1}\sum_{i\leq n}\|Z_i\| \leq \sqrt{\sum_{i\leq n}\|Z_i\|^2/n}$ it follows that

$$\mathbb{P}\{n^{-1}\sum\nolimits_{i\leq n}\|Z_i\| \geq 2\sqrt{d}\,\} \leq e^{-nd/2},$$

a neat little bound that is useful in high-dimensional statistical theory. See Wu and Zhou (2019, Section 9), for example. □

3.7 Binomial

MGF::S:Binomial

The Binomial distribution behaves a little like the Poisson. It is also a prototype for other inequalities involving sums of bounded random variables.

Remember that X has a $\text{Bin}(n, p)$ distribution if

$$\mathbb{P}\{X = k\} = \binom{n}{k} p^k q^{n-k} \qquad \text{for } k = 0, 1, \ldots, n.$$

Here and subsequently I write q for $1 - p$. The distribution has expected value np, variance npq, and $M_X(\lambda) = (q+pe^\lambda)^n$. The random variable $n-X$ has a $\text{Bin}(n, q)$ distribution. Thus

\E@ upper.lower <28>

$$\mathbb{P}\{X \leq np - t\} = \mathbb{P}\{n - X \geq nq + t\}.$$

That is, the lower tail for the $\text{Bin}(n, p)$ corresponds exactly to the upper tail for the $\text{Bin}(n, q)$.

Here is the main result: If $X \sim \text{Bin}(n, p)$ then

$$\begin{aligned}\mathbb{P}\{X \geq np + t\} &\leq \exp\left(-np\hbar(t/np) - nq\hbar(-t/nq)\right)\\ &= \exp\left(-\frac{t^2}{2npq}g_p(x)\right) \qquad \text{for } 0 \leq t \leq nq\end{aligned}$$

\E@ Bin.upper <29>

$$\text{where } g_p(t) := q\psi_{\text{Benn}}\left(\frac{t}{np}\right) + p\psi_{\text{Benn}}\left(\frac{-t}{nq}\right).$$

From equality <28> the companion inequality for the lower tail is

$$\mathbb{P}\{X \leq np - t\} \leq \exp\left(-\frac{t^2}{2npq}g_q(t)\right) \qquad \text{for } 0 \leq t \leq np.$$

It is merely a matter of swapping the roles of p and q.

Remarks.

(i) Note that

$$g_p(t)/(2npq) \to \psi_{\text{Benn}}(t/\theta)/(2\theta) \qquad \text{if } n \to \infty \text{ and } np \to \theta \in \mathbb{R}^+.$$

Not surprisingly we then recover the MGF tail bounds for the POISSON(θ) distribution.

(ii) This $g_p(t)$ is the same function as the $g(t, n, p)$ in Section 2.5, which derived sharp approximations for $\mathbb{P}\{X = k\}$ by means of the Stirling approximation: for $k = np + t$,

$$\mathbb{P}\{X = k\} = \frac{\exp\left[-t^2 g_p(t)/(2npq) + O\left(k^{-1} + (n-k)^{-1}\right)\right]}{\sqrt{2\pi n(p + t/n)(q - t/n)}}.$$

As happened with the $N(0, 1)$, the MGF method captures the main term in the exponent but misses the square root term in the denominator.

(iii) The bound $t \le nq$ is not really necessary, because $\mathbb{P}\{0 \le X \le n\} = 1$. It merely serves to ensure that $t/(np)$ and $-t/(nq)$ are both ≥ -1, so that we don't have to worry about $\psi_{\text{Benn}}()$ taking the value $+\infty$. We could also let the the definition of $L_{X-np}(t)$ take care of the difficulty by having it take the value $+\infty$ when $t < -np$ or $t > nq$. Compare with the calculation for the Poisson in Section 3.5.

(iv) The Taylor expansion $\hbar(x) = x^2/2! - x^3/3! + O(x^4)$ gives

$$np\hbar(t/np) + nq\hbar(-t/nq) = \frac{t^2}{2npq} - \frac{t^3(q-p)}{6(npq)^2} + O(t^4) \qquad \text{for } t \text{ near } 0,$$

which agrees with the calculations in Section 3.2.

(v) As explained in Section 2.5, the convexity of ψ_{Benn} gives the inequality

$$g_p(t) \ge \psi_{\text{Benn}}\left(\frac{qt}{np} - \frac{pt}{nq}\right) = \psi_{\text{Benn}}\left(\frac{t(q-p)}{npq}\right) \ge 1 \qquad \text{if } p \ge 1/2.$$

Thus, if $p \ge 1/2$, the upper tail is less than $\exp\left(-t^2/(2npq)\right)$, a clean subgaussian bound with scale parameter $\sqrt{npq}$. (As commented in Section 2.5, this subgaussian fact can also be interpreted as a skewness effect.) If $p < 1/2$ the upper tail is still subgaussian (because $\psi_{\text{Benn}}(-t/(nq)) \ge 1$) but with a larger scale parameter $\sqrt{nq}$.

Proof (of inequality <29>). From <5>, for $t \ge 0$ we need to find the supremum over $\mathbb{R}^+$ of

$$\mathcal{L}(\lambda) := (t + np)\lambda - n\log(q + pe^{\lambda}),$$

which has derivative

$$\mathcal{L}'(\lambda) = (t + np) - npe^{\lambda}/(q + pe^{\lambda}).$$

If $t = nq$ then $\mathcal{L}'(\lambda) > 0$ on $\mathbb{R}^+$ and $\mathcal{L}(\lambda) = n \log\left(e^{\lambda}/(q + pe^{\lambda})\right)$, so that the supremum $n \log(1/p)$ is approached as $\lambda \to \infty$. The final bound then reduces to $\mathbb{P}\{X \geq n\} \leq p^n$, which is actually true with equality.

If $0 \leq t < nq$ then the maximum is achieved at the λ for which $\mathcal{L}'(\lambda) = 0$, that is, when $(t + np)(q + pe^{\lambda}) = npe^{\lambda}$. The algebra is then simplified a trifle if we write z_1 for $t/(np)$ and z_2 for $t/(nq)$. The equation becomes

$$(1 + z_1)q = e^{\lambda}\left[1 - p(1 + z_1)\right] = e^{\lambda} q(1 - z_2)$$

because $p(1 + z_1) + q(1 - z_2) = 1$. That is, the maximizing λ is given by $e^{\lambda} = (1 + z_1)/(1 - z_2)$ and

$$\begin{aligned}
\Lambda(t) &= np(1 + z_1) \log\left(\frac{1 + z_1}{1 - z_2}\right) - n \log\left(q + p(1 - z_1)/(1 - z_2)\right) \\
&= np(1 + z_1)\log(1 + z_1) - np(1 + z_1)\log(1 - z_2) \\
&\quad - n \log\left(q(1 - z_2) + p(1 + z_1)\right) + n \log(1 - z_2) \\
&= np\left(\mathbb{h}(z_1) + z_1\right) - \log(1) + n\left(1 - p(1 + z_1)\right)\log(1 - z_2) \\
&= np\left(\mathbb{h}(z_1) + t/(np)\right) + nq\left(\mathbb{h}(-z_2) - t/(nq)\right),
\end{aligned}$$

□ which simplifies to the $np\mathbb{h}(z_1) + nq\mathbb{h}(-z_2)$ for the first line of <29>.

Now let me move a little beyond the Binomial to show that there are several other distributions, with the same expected value as the $\text{Bin}(n, p)$, that share the tail bounds for the Binomial. In this Chapter these bounds are derived by means of pointwise inequalities for MGFs. As such they leave open the question of whether analogous inequalities would also hold for the exact tail probabilities, not just their MGF-derived upper bounds. Chapter 4 will return to this question

MGF::PoisBin <30> **Example.** Suppose $S \sim \text{PBin}(p_1, \ldots, p_n)$, that is, S is a sum of independent random variables $Y_1 + \cdots + Y_n$, with $Y_i \sim \text{Ber}(p_i)$ for possibly different p_i's. Define $\overline{p} = n^{-1}\sum_i p_i$. Then

$$M_S(\lambda) = \prod\nolimits_{i \leq n}\left(q_i + p_i e^{\lambda}\right) = \exp\left(\sum\nolimits_{i \leq n} \log(q_i + p_i e^{\lambda})\right).$$

Concavity of the log() function shows that

\E@ PB.MGF <31> $$n^{-1}\sum\nolimits_{i \leq n} \log(q_i + p_i e^{\lambda}) \leq \log\left(n^{-1}\sum\nolimits_{i \leq n}(q_i + p_i e^{\theta})\right) = \log\left(\overline{q} + \overline{p}e^{\theta}\right).$$

Thus $M_S(\theta) \le M_W(\theta)$ where $W \sim \text{BIN}(n, \overline{p})$ and $\overline{q} = 1 - \overline{p}$. It follows that

$$\mathbb{P}\{S \ge n\overline{p} + x\} \le \exp\left(-\frac{x^2}{2n\overline{p}\overline{q}} g_{\overline{p}}(x)\right) \qquad \text{for } 0 \le x \le n\overline{q},$$

□ with a similar bound for the lower tail.

The convexity idea from the previous Example can be pushed even further.

MGF::Hoeffding <32> **Example.** Suppose $T = Y_1 + \cdots + Y_n$, a sum of independent random variables Y_i with $0 \le Y_i \le 1$ and $\mathbb{P}Y_i = p_i$ for each i, and $n\overline{p} = \sum_{i=1}^n p_i$. By convexity of the exp() function,

$$e^{\lambda Y_i} \le (1 - Y_i) + Y_i e^{\lambda} \qquad \text{for each real } \lambda.$$

The inequality holds for all possible realizations of Y_i. Equality is achieved at $Y_i \in \{0, 1\}$. In particular, equality holds when $Y_i \sim \text{BER}(p_i)$, as in the previous Example. Take expectations.

$$\mathbb{P}e^{\lambda Y_i} \le (1 - p_i) + p_i e^{\lambda} = q_i + p_i e^{\lambda} \qquad \text{for each real } \lambda.$$

By independence,

$$M_T(\lambda) = \prod\nolimits_{i \le n} \mathbb{P}e^{\lambda Y_i} \le \prod\nolimits_{i \le n} \left(q_i + p_i e^{\lambda}\right) = M_S(\lambda) \le M_W(\lambda),$$

with $S \sim \text{PBIN}(p_1, \ldots, p_n)$ and $W \sim \text{BIN}(n, \overline{p})$, as in the previous Example. Thus

$$\mathbb{P}\{T \ge n\overline{p} + x\} \le \exp\left(-\frac{x^2}{2n\overline{p}\overline{q}} g_{\overline{p}}(x)\right) \qquad \text{for } 0 \le x \le n\overline{q},$$

□ a result due to Hoeffding (1963, Theorem 1).

3.8 Sampling and the hypergeometric

MGF::S:hypergeometric

Both Example <30> and Example <32> involved sums of independent random variables. The MGF approach can also work when there is dependence between the summands, although the argument becomes a little more delicate.

MGF::hyper <33>

Example. Suppose $U = \{u_1, \ldots, u_N\}$ is a finite set, an urn if you like to think that way. In that interpretation the u_i's are the balls. Suppose exactly R of the balls are colored red and the other $B = N - R$ are colored black. If n balls are sampled without replacement then each subset of U with size n has probability $1/\binom{N}{n}$ of being selected and the number of red balls T_n in the sample has a hypergeometric distribution, $\text{HYPER}(n, R, B)$, meaning that

$$\mathbb{P}\{T_n = k\} = \binom{R}{k}\binom{B}{n-k}/\binom{N}{n}$$

for each nonnegative integer k such that $k \le R$ and $n - k \le B$.

If the sampling is carried out with replacement then the number of red balls in the sample, S_n, has a $\text{BIN}(n, p)$ distribution, where $p = R/N$.

Elementary calculations (Pitman, 1993, Section 3.6) show that

$$\begin{aligned} \mathbb{P}T_n &= \mathbb{P}S_n = np \\ \text{var}(T_n) &= np(1-p)(N-n)/(N-1) < \text{var}(S_n) = np(1-p). \end{aligned}$$

If n is much smaller than N then there is actually not much difference between $\text{HYPER}(n, R, B)$ and $\text{BIN}(n, p)$: if a ball is selected then returned to the urn, it is unlikely to be selected again if n/N is very small. If n/N is not so small then, judging by the variances, $\text{HYPER}(n, R, B)$ is more concentrated around np than $\text{BIN}(n, p)$. A beautiful result by Hoeffding (1963, Section 6) adds some precision to this intuition. He showed that for each convex function f on the real line,

\E@ Urn.Jensen <34>

$$\mathbb{P}f(T_n) \le \mathbb{P}f(S_n).$$

In particular, the choice $f(x) = e^{\lambda x}$ shows that $M_{T_n}(\lambda) \le M_{S_n}(\lambda)$ for all real λ. Any tail bound for the hypergeometric obtained via the MGF argument must therefore be smaller than the corresponding MGF tail bound for the Binomial.

To be more precise, Hoeffding's result didn't involve red balls and black balls. It worked for every function $g : U \to \mathbb{R}$. (The special case where $g(u_i) = 1$ for a red ball an $g(u_i) = 0$ for a black ball get us back to the hypergeometric.) That is, we can take $X_1, \ldots, X_n$ to be a sample from U without replacement and $Y_1, \ldots, Y_n$ to be a sample with replacement. If we define $T_n := \sum_{i \le n} g(X_i)$ and $S_n := \sum_{i \le n} g(Y_i)$ then inequality <34> will still hold for every convex f.

I had some trouble digesting Hoeffding's proof. Even after working through the details I could not have explained to anyone why the method

worked. Subsequently I stumbled on a proof by Le Cam (1986, page 534), which involved a much more intuitive explanation, reducing everything to Jensen's inequality. Unfortunately I again had some trouble convincing myself that all the intuitions were completely watertight, so I wrote out the following rather more pedantic account based Le Cam's idea. For technical details see Problem [11].

Here is the key idea. Suppose $Y = (Y_1, Y_2, \dots)$ is obtained by sampling repeatedly with replacement from U. With probability one each member of U appears infinitely often in the Y sequence. If we discard all except the first appearance of each u in U from the Y sequence then we are left with a random permutation, $(X_1, \dots, X_N)$ of U; and $X_1, \dots, X_n$ forms a sample of size n taken without replacement from U.

The sequence $(Y_1, Y_2, \dots)$ will contain repeats, which can be represented as a sequence $\mathcal{C}(Y) = (\mathcal{C}_1(Y), \mathcal{C}_2(Y), \dots)$ of symbols from a set of 'code-words' $\mathbb{B} = \{\boxed{j} : 1 \le j \le N\}$, by the following procedure. Think of $\mathbb{B}$ as ordered: $\boxed{1} < \boxed{2} < \cdots < \boxed{N}$. The code $\mathcal{C}(Y)$ always starts with $\boxed{1}$. If $Y_2 = Y_1$ then $\mathcal{C}_2(Y) = \boxed{1}$, otherwise $\mathcal{C}_2(Y) = \boxed{2}$. And so on. In general, if a Y_i repeats an earlier Y_j then $\mathcal{C}_i(Y) = \mathcal{C}_j(Y)$; if Y_i is different from all previous Y_j's then it receives the smallest unused code symbol. For example, here is how it works for a typical Y:

Y:	u_7	u_3	u_9	u_7	u_2	u_3	u_3	u_{185}	$\dots$
X:	u_7	u_3	u_9		u_2			u_{185}	$\dots$
$\mathcal{C}(Y)$:	$\boxed{1}$	$\boxed{2}$	$\boxed{3}$	$\boxed{1}$	$\boxed{4}$	$\boxed{2}$	$\boxed{2}$	$\boxed{5}$	$\dots$

You should ignore the gaps in the X-vector; I inserted them just to align each X_j with its first appearance in the Y sequence. The corresponding positions in $\mathcal{C}(Y)$ contain a repetition of an earlier code symbol. For example, the second u_7 in the Y sequence has a gap in X and is coded as $\boxed{1}$ because $Y_1 = u_7$.

Together, X and $\mathcal{C}(Y)$ allow us to reconstruct Y: for $i = 1, \dots, N$ replace each $\boxed{i}$ in $\mathcal{C}(Y)$ by X_i, the ith element of X (ignoring the $\sqcup$ characters). More concisely, $Y_j = X_{\mathcal{C}(Y_j)}$, provided we ignore the little box around the code symbol.

I claim that X and $\mathcal{C}(Y)$ are independent.

Remark. Initially I thought the independence was obvious: knowledge of the pattern tells us nothing about the order in which the elements of U are first observed. For example, if $\mathcal{C}(Y) = (\boxed{1}, \boxed{2}, \boxed{3}, \boxed{1}, \dots)$ then we know that Y_1, Y_2, Y_3 are different elements of U and $Y_4 = Y_1$ but we

have no information about which three elements of U were involved. Then I began to worry that this assertion was a bit too hand-waving. It took me a while to come up with the more rigorous argument given in Problem [11].

Now back to <34>. Remember that $S_n = g(Y_1) + \dots g(Y_n)$ and $T_n = g(X_1) + \dots g(X_n)$. The sum S_n can be re-expressed using the counts

$$N_n(j) = \text{number of times } \boxed{j} \text{ appears amongst } \mathcal{C}(Y_1), \dots, \mathcal{C}(Y_n).$$

For example, if $n = 6$ and $Y = (u_7, u_3, u_9, u_7, u_2, u_3, \dots)$ then

$$(X_1, \dots, X_4) = (u_7, u_3, u_9, u_2) \quad \text{AND} \quad \mathcal{C}(Y) = (\boxed{1}, \boxed{2}, \boxed{3}, \boxed{1}, \boxed{4}, \boxed{2} \dots),$$

so that $N_6(1) = N_6(2) = 2$ and $N_6(j) = 1$ for $j = 3, 4$, which gives

$$g(Y_1) + \cdots + g(Y_6) = 2g(X_1) + 2g(X_2) + g(X_3) + g(X_4).$$

Notice that we only need the counts up to $j = 6$, at most, because $(Y_1, \dots, Y_6)$ can involve at most 6 different elements $X_1, \dots, X_6$ of U. In general,

$$S_n = g(Y_1) + \cdots + g(Y_n) = \sum\nolimits_{j=1}^{n} N_n(j) g(X_j)$$

and

\E@ Y.rep <35>
$$\mathbb{P} f\Big(S_n\Big) = \mathbb{P} f\left(\sum\nolimits_{j=1}^{n} N_n(j) X_j\right)$$

Unfortunately, the final expression is not symmetric in $X_1, \dots, X_n$; it is hard to see how it is related to $\mathbb{P} f(T_n)$. My method for determining patterns broke the symmetry but it can be restored using a sneaky trick. As the $N_n(j)$'s depend only on $\mathcal{C}(Y)$ they are independent of X. We could replace $(X_1, \dots, X_n)$ by any other random sequence $(\widetilde{X}_1, \dots, \widetilde{X}_n)$ that is independent of $\mathcal{C}(Y)$ and has the same distribution as $(X_1, \dots, X_n)$. For example, for any fixed permutation σ of $[\![n]\!] := \{j \in \mathbb{N} : j \le n\}$ we could use $(X_{\sigma(1)}, \dots, X_{\sigma(n)})$:

$$\mathbb{P} f\Big(S_n\Big) = \mathbb{P} f\left(\sum\nolimits_{j=1}^{n} N_n(j) X_{\sigma(j)}\right) \qquad \text{for each permutations } \sigma \text{ of } [\![n]\!].$$

We can even average out over the uniform distribution $\mathbb{Q}$ on the set of all permutations of $[\![n]\!]$ then use Jensen to take the $\mathbb{Q}$ integral inside the convex function:

$$\mathbb{P} f\Big(S_n\Big) = \mathbb{Q}^\sigma \mathbb{P} f\left(\sum\nolimits_{j=1}^{n} N_n(j) X_{\sigma(j)}\right) \ge \mathbb{P} f\left(\mathbb{Q}^\sigma \sum\nolimits_{j=1}^{n} N_n(j) X_{\sigma(j)}\right)$$

From the facts that $\sum_{j=1}^{n} N_n(j) = n$ and

$$\mathbb{Q}^\sigma g(X_{\sigma(j)}) = n^{-1} \sum\nolimits_{i=1}^{n} g(X_i) = n^{-1} T_n \qquad \text{for } 1 \le j \le n$$

□ it now follows that $\mathbb{P} f(S_n) \ge \mathbb{P} f(T_n)$, as asserted.

3.9 Problems

MGF::S:Problems

For Problems [1] through [7], the function $\phi(x)$ denotes the $N(0,1)$ density and $\bar{\Phi}(x) = \int_x^\infty \phi(t)\,dt$; the functions $\mathcal{R}(\cdot)$ and $\rho(\cdot)$ and $r(\cdot)$ are defined on $\mathbb{R}$ by $1/\rho(x) = \mathcal{R}(x) = \bar{\Phi}(x)/\phi(x)$ and $r(x) = \rho(x) - x$.

MGF::P:Laplace [1] Inequality <11> is just the initial part of a sequence of upper and lower bounds for $\mathcal{R}(x)$, which are apparently due to Laplace (see Notes). Each bound is of the form $p(1/x)$ with p a polynomial.

(i) Show that $p(1/x) > \mathcal{R}(x)$ for all $x > 0$ if

\E@ Mill.upper <36> $$-\frac{d}{dt}\left(p(1/t)\phi(t)\right) > \phi(t) \qquad \text{for all } t > 0$$

and $p(1/x) < \mathcal{R}$ for all $x > 0$ if

\E@ Mill.lower <37> $$-\frac{d}{dt}\left(p(1/t)\phi(t)\right) < \phi(t) \qquad \text{for all } t > 0$$

Hint: $\int_x^\infty$.

(ii) Show that<36> holds if and only if $p(t) + t^3 p'(t) > t$ for all $t > 0$. Characterize <37> by the reverse inequality.

(iii) Define a sequence of monomials by $\Delta_0(t) = t$ and $\Delta_k(t) = -t^3\Delta'_{k-1}(t)$ for $k \geq 1$. Show that

$$\Delta_k(t) = (-1)^k a_k t^{2k+1} \qquad \text{where } a_k = 1 \times 3 \times \cdots \times (2k-1).$$

(iv) Define $p_k(t) = \sum_{i=0}^k \Delta_i(t)$. Show that $p_k(t) + t^3 p'_k(t) = t + \Delta_{k+1}(t)$.

(v) Conclude that $p_k(1/x) > \mathcal{R}(x) > p_{k+1}(1/x)$ for each even k. For example, for $k = 1$ and $k = 2$ we have, for all $x > 0$,

$$x^{-1} > \mathcal{R}(x) > x^{-1} - x^{-3}$$
$$x^{-1} - x^{-3} + 3x^{-5} > \mathcal{R}(x) > x^{-1} - x^{-3} + 3x^{-5} - 15x^{-7}$$

(vi) From the inequality for $k = 2$ deduce that $xr(x) \to 1$ as $x \to \infty$.

MGF::P:rho.facts [2] Here are the basic facts about $\rho(\cdot)$ and $r(\cdot)$.

(i) Suppose $Z \sim N(0,1)$. Show that

$$\mathcal{R}(x) = \int_0^\infty \phi(x+t)/\phi(x)\,dt = \int_0^\infty e^{-xt-t^2/2}dt = \sqrt{\pi/2}\,\mathbb{P}e^{-x|Z|}.$$

(ii) Using Section 2.3, show that $L(x) := \log \mathcal{R}(x) = -\log \rho(x)$ is a strictly decreasing, convex function. Deduce that $\rho(x) = e^{-L(x)}$ is strictly increasing with $\rho(x) \to 0$ as $x \to -\infty$ and $\rho(x) \to \infty$ as $x \to \infty$. Also $\log \rho(x) = -x^2/2 - \log \sqrt{2\pi} - \log \bar{\Phi}(x)$ is concave.

(iii) Using the bounds from Problem [1], show that $\rho(x) > x$ and $r(x) > 0$ for all x. (Note that $\rho(x) > x$ is trivially true for $x \le 0$.) Also show that $xr(x) \to 1$ as $x \to \infty$.

(iv) Show that $\log(\bar{\Phi})$ has derivative $-\rho$ so that $\rho'(x)/\rho(x) = d \log \rho(x)/dx = \rho(x) - x$. That is, $\rho'(x) = \rho(x)r(x)$ for all x. From the concavity of $\log \rho$ deduce that r is a decreasing function.

(v) (Sampford, 1953) Show that $\gamma(x) := \rho''(x)/\rho(x) = 2r(x)^2 + xr(x) - 1$. Show that

$$\gamma'(x) = (4r(x) + x)\, r'(x) + r(x) = 2r(x)r'(x) + \rho(x)\gamma(x) < \rho(x)\gamma(x).$$

Argue as follows to show that $\gamma(x) > 0$ for all $x \in \mathbb{R}$, which implies that ρ is strictly convex. Suppose there were an x_0 for which $\gamma(x_0) \le 0$. By the preceding argument, $\gamma'(x_0)$ would be < 0. There would therefore be some $\delta > 0$ and $x_1 > x_0$ at which $\gamma(x_1) < -\delta$. By part (iii), $\gamma(x) \to 0$ as $x \to \infty$. For some finite K there would exist some $K > x_1$ for which $|\gamma(x)| < \delta$ for $x > K$. The difefrentiable function γ would achieve its minimum value on $[x_0, \infty)$ at some point x_2 in $[x_0, K]$ at which $0 > -\delta \ge \gamma(x_2)$ and $\gamma'(x_2) = 0$, a contradiction.

MGF::P:half [3] Show that $\bar{\Phi}(x) \le \frac{1}{2} e^{-x^2/2}$ for $x \ge 0$. Hint: From Problem [2](i) we have $\mathcal{R}(0) > \mathcal{R}(x)$.

MGF::P:Px [4] For each x show (by means of integration-by-parts) that $\int_x^\infty t\phi(t)\, dt = \phi(x)$ and $\int_x^\infty t^2 \phi(t)\, dt = x\phi(x) + \bar{\Phi}(x)$. Let P_x be the probability measure with density $p_x(t) = \phi(t)\{t \ge x\}/\bar{\Phi}(x)$ with respect to Lebesgure measure on the real line. Show that the variance of P_x is $1 - \rho(x)r(x)$. Deduce that $\rho(x)r(x) < 1$ for all x.

MGF::P:Birnbaum [5] (Birnbaum, 1942). Use Cauchy-Schwarz and facts from Problem [4] to show that

$$\phi(x)^2 = \left(\int_x^\infty t \sqrt{\phi(t)} \sqrt{\phi(t)}\, dt \right)^2 \le \left(x\phi(x) + \bar{\Phi}(x) \right) \bar{\Phi}(x).$$

Deduce that $1 \le (x + \mathcal{R}(x))\, \mathcal{R}(x) = \left(\mathcal{R}(x) + \frac{1}{2}x \right)^2 - x^2/4$, which implies $\rho(x) \le \left(x + \sqrt{x^2 + 4} \right)/2$.

MGF::P:Sampford

[6] (Sampford, 1953) Let $\gamma(x) = 2r(x)^2 + xr(x) - 1$, as in Problem [2](v). Remember that $\gamma(x) > 0$ for all $x \in \mathbb{R}$. Argue that $r(x)$ cannot belong to the closed interval $I_x := \{t \in \mathbb{R} : 2t^2 + xt - 1 \le 0\}$, which has endpoints $\left(-x \pm \sqrt{x^2+8}\right)/4$. Deduce that $r(x) > (-x + \sqrt{x^2+8})/4 = 2/(x + \sqrt{x^2+8})$ and $\rho(x) > (3x + \sqrt{x^2+8})/4$. Note: $r(x) > 0$.

MGF::P:expected.max

[7] Suppose $Z_1, \ldots, Z_n$ are random variables, each distributed $N(0,1)$ but, for the moment, not necessarily independent. Define $M_n = \max_{i \le n} Z_i$.

(i) Even without independence the MGF approach also gives an upper bound a_n for the expected value of M_n, via Jensen's inequality: for each $\lambda > 0$,

$$\exp(\lambda \mathbb{P} M_n) \le \mathbb{P} e^{\lambda M_n} = \mathbb{P} \max_i e^{\theta Z_i} \le \sum_i \mathbb{P} e^{\lambda Z_i} = n e^{\lambda^2/2}.$$

Deduce that $\mathbb{P} M_n \le \inf_{\lambda > 0} \left(\log n + \lambda^2/2\right)/\lambda = a_n = \sqrt{2\log(n)}$. The case where $Z_i = Z_1$ for all i shows that the bound is not sharp in general.

(ii) If the Z_i's are independent, show that $\mathbb{P} M_n \ge a_n - c\log(a_n)/a_n$ for some constant c, if n is large enough. First show that

$$M_n \ge \max_{i \le n} Z_i^+ - \left(\sum_{i \le n} |Z_i|\right) \{M_n \le 0\}$$

so that $\mathbb{P} M_n \ge \mathbb{P} \max_{i \le n} Z_i^+ - n\mathbb{P}|Z_1|/2^{n-1}$. Then argue that

$$O(n/2^{n-1}) + \mathbb{P} M_n \ge \mathbb{P} \max_{i \le n} Z_i^+ = \int_0^\infty \mathbb{P}\{M_n > t\}\, dt \ge x_n \mathbb{P}\{M_n > x_n\}.$$

Look at Example <13> for a way to choose x_n.

MGF::P:max.abs.normals

[8] Suppose $Z_1, \ldots, Z_n$ are independent random variables, each distributed $N(0,1)$.

(i) Show that $\mathbb{P}\{\max_{i \le n} |Z_i| \le x_n\} = \left(1 - 2\bar{\Phi}(x_n)\right)^n$.

(ii) Mimic the argument from Example <13> to deduce that $\max_{i \le n} |Z_i|$ concentrates near a_n.

MGF::P:std.exp

[9] Suppose X has a standard exponential distribution.

(i) Show that $\mathbb{P}\{X \ge x\} = e^{-x}$ for all $x \ge 0$ and $\mathbb{P} X = 1$.

(ii) Show that the method from Section 3.1 gives $\mathbb{P}\{X \ge x\} \le (ex)e^{-x}$ for $x \ge 1$.

(iii) What bound does the method give for $0 \le x < 1$?

MGF::P:union.indep [10] Suppose $A_1, \ldots, A_n$ are independent events with $\sum_i \mathbb{P}A_i = \epsilon$, for a small ϵ. Show that

$$\mathbb{P} \cup_i A_i = 1 - \exp\left[\sum\nolimits_i \log(1 - \mathbb{P}A_i)\right] \geq 1 - e^{-\epsilon} = \epsilon - O(\epsilon^2).$$

MGF::P:indep.code [11] Here is a rigorous way to establish independence of X and $\mathcal{C}(Y)$ in Example <33>. Notation (derived from Section 3.8):

a) Regard Y as the identity map (that is, $Y(y) = y$) on $U^{\mathbb{N}}$ equipped with its product sigma-field and product measure $\mathbb{P} = \nu^{\mathbb{N}}$, where ν denotes the uniform distribution on U.

b) $\mathbb{B} = \{\boxed{i} : i = 1, \ldots, N\}$, the code symbols. Regard $\mathcal{C}$ as a measurable map from $U^{\mathbb{N}}$ into the product space $\mathbb{B}^{\mathbb{N}}$ (equipped with its product sigma-field).

c) $\mathcal{W}$ = the set of all permutations of U. (Thus $|\mathcal{W}| = N!$.) If $x = (x_1, \ldots, x_N) \in \mathcal{W}$ and $\boxed{i} \in \mathbb{B}$, interpret $x_{\boxed{i}}$ to mean x_i.

d) Treat $X = X(y)$ as the map from $U^{\mathbb{N}}$ into the set $\mathcal{W}$ that is defined by discarding repetitions of each y_i after its first appearance in y.

e) For $y = (y_1, y_2, \ldots) \in U^{\mathbb{N}}$ and $B = (b_1, b_2, \ldots) \in \mathbb{B}^{\mathbb{N}}$ define $\mathcal{C}(y) \,[\!]\, n = (\mathcal{C}_1(y), \ldots, \mathcal{C}_n(y))$ and $B \,[\!]\, n = (b_1, \ldots, b_n)$.

The argument:

(i) Show that the distribution of X is uniform on $\mathcal{W}$. Deduce that

$$\mathbb{P}\{X_j = x_j, \ldots, X_N = x_N\} = \frac{1}{N} \times \frac{1}{N-1} \times \frac{1}{N-j+1} = \frac{1}{(N)_j}$$

for each $x \in \mathcal{W}$ and $1 \leq j \leq N$. (Exchangeability helps.)

(ii) Show that the distribution of $\mathcal{C}(Y)$ concentrates on the set $\mathcal{P}$ of all *feasible* B's in $\mathbb{B}^{\mathbb{N}}$, that is, those B that start with $\boxed{1}$ and for all i, j with $1 \leq j < i \leq N$ the codeword $\boxed{j}$ first appears in B before $\boxed{i}$.

(iii) Suppose $B \in \mathcal{P}$ and $B \,[\!]\, n$ uses only the k code symbols $\boxed{1}, \ldots, \boxed{k}$. Show that

$$\mathbb{P}\{\mathcal{C}(Y) \,[\!]\, n = B \,[\!]\, n\} = \frac{N(N-1)\ldots(N-k+1)}{N^n} = \frac{(N)_k}{N^n}.$$

Hint: Each repeat of a codeword corresponds to an event that has probability $1/N$ and the first appearance of codeword $\boxed{j}$ indicates a selection from a set $N - j + 1$ elements from U.

(iv) Suppose $x = (x_1, \ldots, x_N) \in \mathcal{W}$ and $B \in \mathcal{P}$, with $B \,[\!]\, n$ using only the k code symbols $\boxed{1}, \ldots, \boxed{k}$. Justify the following assertions.

Define

$$A = \{y : \mathcal{C}(y) \,[\!]\, n = B \,[\!]\, n, \quad X_i = x_i \text{ for } i = 1, \ldots, k \}.$$

The assumption about B means that $y_1, \ldots, y_n$ select only k distinct elements of U, namely $U_1 := \{x_i : i = 1, \ldots, k\}$, with first selections occurring in the order $(x_1, \ldots, x_k)$. Thus

$$\mathbb{P}A = \mathbb{P}\{y : y_i = x_{b_i} \text{ for } i = 1, \ldots, n \} = (1/N)^n.$$

Conditional on the occurrence of A, the remaining observations $y_{n+1}, y_{n+2}, \ldots$ are left to select each element of $U_2 := \{x_i : n + 1 \le i \le N\}$. If we also require $X_i = x_i$ for $k + 1 \le i \le N$ then we have specified the order of first selections of the elements of U_2, namely $(x_{k+1}, \ldots, x_N)$. Thus

$$\mathbb{P}\{X_i = x_i \text{ for } k + 1 \le i \le N \mid A\} = \frac{1}{N-k} \times \frac{1}{N-k-1} \times \cdots \times \frac{1}{1} = \frac{1}{(N-k)!}.$$

Combining these two results we get

$$\begin{aligned}
&\mathbb{P}\{y : \mathcal{C}(y) \,[\!]\, n = B \,[\!]\, n,\, X = (x_1, \ldots, x_N)\} \\
&\quad = \mathbb{P}\left(A \cap \{X_i = x_i \text{ for } k + 1 \le i \le N \}\right) \\
&\quad = (1/N)^n \times \frac{1}{(N-k)!} = \frac{N(N-1)\ldots(N-k+1)}{N^n} \times \frac{1}{N!} \\
&= \mathbb{P}\{\mathcal{C}(Y) \,[\!]\, n = B \,[\!]\, n\} \times \mathbb{P}\{X = x\}.
\end{aligned}$$

It follows that $\mathcal{C}(Y)$ and X are independent.

3.10 Notes

MGF::S:Notes

Bennett (1962) and Hoeffding (1963) are good sources for a host of exponential inequalities. Massart (2003, Chapter 2) and Boucheron, Lugosi, and Massart (2013, Chapter 2) persuaded me that it is a good idea to have the relevant ideas collected together in one place, rather than deriving them on an ad hoc basis.

The following lengthy discussion is aimed particularly at younger (or not so young) researchers who rely on Wikipedia for their probability theory and their history.

Many authors seem to credit Chernoff (1952) with the moment generating trick in <1>, even though the idea is obviously much older. Here is what Chernoff actually said. He first noted (page 494) that Cramér (1938) had already established excellent results for sums of independent random variables using the MGF method.

> **Remark.** Cramér's 1938 paper summarized asymptotic approximations to the tail probabilities for a sum of independent random variables, rather than bounds on those tail probabilities; details appeared in Chapter 7 of Cramér (1937). See Cramér (1976, Section 4.9) for comments about the conference where he presented the 1938 paper.

Chernoff (page 494) continued:

> Since the results of Cramér are extremely more powerful tha[n] we require here and the (finite) existence of third order moments is not necessary for the results that we desire, we shall state and briefly outline a proof of Theorem 1. Before doing this we shall first formally state some notation and lemmas which we shall use throughout this paper. These lemmas state known results which are rather obvious, depending mainly on Lebesgue's Theorem on integration of monotone sequences [reference to the Saks book].

Note the words 'state known results'.

> **Remark.** In the following paragraphs I have slightly modified Chernoff's notation, to make it agree with my notation.

He then proceeded (page 495) to list some properties of the MGF for $S_n = X_1 + \cdots + X_n$, a sum of independent random variables, each distributed like X. He defined $m(a) = \inf_{t \in \mathbb{R}} e^{-at} M_X(t)$. After noting some facts about derivatives of M_X, he came to his Theorem 1. He listed several inequalities, such as

$$\mathbb{P}\{S_n \geq na\} \leq [m(a)]^n \qquad \text{if } \mathbb{P}X \leq a < \infty.$$

Chernoff provided (page 496) a brief sketch of the proof of his Theorem 1. He began with the 'extended Tchebycheff inequality' for $a = 0 \leq \mathbb{P}X$,

$$\mathbb{P}e^{tS_n} = [M_X(t)]^n \geq \mathbb{P}\{S_n \leq 0\} \qquad \text{for } t \leq 0,$$

citing the German original of Kolmogorov (1933). (See page 42 in the English translation.) Kolmogorov gave no source, although the 1927 edition of

Bernstein's probability book was listed in his Bibliography. Chernoff then went on to derive properties of the minimum that were of interest to him.

Undoubtedly there was much in Chernoff's paper that was new and highly influential on the subsequent statistical literature. The idea of bounding a tail probability by a minimization involving the MGF was not new. From the pen of Chernoff himself (DasGupta, 2004, page 3):

> That semester, two topics that arose from the ONR project gave rise to two papers that I wrote and of which I was very proud. They pointed to a direction in optimal experimental design on which I spent much time later. Part of one of these papers involved finding asymptotic upper and lower bounds on the probability that the mean of a sample of independent identically distributed random variables would exceed a certain constant. This paper represented the first application of large deviation theory to a statistical problem. Cramer had derived a much more elegant result in 1938, of which I had been ignorant. My result, involving the infimum of a moment generating function, was less elegant and less general than the Cramer result, but did not require a special condition that Cramer required. Also, my proof could be described as crudely beating the problem to death. Herman claimed that he could get a lower bound much easier. I challenged him, and he produced a short Chebyshev Inequality type proof, which was so trivial that I did not trouble to cite his contribution.
>
> What a mistake! It seems that Shannon had incorrectly applied the Central Limit theorem to the far tails of the distribution in one of his papers on Information theory. When his error was pointed out, he discovered the lower bound of Rubin in my paper and rescued his results. As a result I have gained great fame in electrical engineering circles for the Chernoff bound which was really due to Herman. One consequence of the simplicity of the proof was that no one ever bothered to read the original paper of which I was very proud. For years they referred to Rubin's bound as the Chernov bound, not even spelling my name correctly. I once had the pleasure of writing to a friend who sent me a copy of a paper improving on the Chernov bound, that I was happy that my name was not associated with such a crummy bound. For many years, electrical engineers have come to me and told me that I saved their lives, because they were able to describe

> the bound on their preliminary doctoral exams. Fortunately for me, my lasting fame, if any, will depend, not on Rubin's bound, but on Chernoff faces.

I think it is very clear that Chernoff was not claiming credit for the whole of the MGF method. Perhaps it was just Chernoff's superlatively clear style that has persuaded some later researcher to call it the Chernoff bound. Maybe someone should modify the Wikipedia entry (copied 28 July 2020):

> In probability theory, the Chernoff bound, named after Herman Chernoff but due to Herman Rubin,[1] gives exponentially decreasing bounds on tail distributions of sums of independent random variables. It is a sharper bound than the known first- or second-moment-based tail bounds such as Markov's inequality or Chebyshev's inequality, which only yield power-law bounds on tail decay. However, the Chernoff bound requires that the variates be independent—a condition that neither Markov's inequality nor Chebyshev's inequality require, although Chebyshev's inequality does require the variates to be pairwise independent.
>
> It is related to the (historically prior) Bernstein inequalities and to Hoeffding's inequality.

ps. The MGF method does not require sums of independent random variables. That case just happens to be the situation that interested Chernoff in 1952.

Now for some more reliable history. In a paper celebrating Bernstein's eightieth birthday, Kolmogorov and Sarmanov (1960) wrote:

> 3. Beginning in 1921 Sergei Natanovich published a number of papers dealing with various special problems in the application of probability theory ... and in 1927 appeared the first edition of the fundamental text "The Theory of Probability", which was reprinted with large supplements in 1934 and 1946. At the mathematical congresses in Moscow (1927) and Zürich (1932) Sergei Natanovich delivered long survey reports on the problems of probability theory. We ... emphasize that at this time such a wide range of work on all the fundamental theoretical and applied problems of probability theory was a totally new thing. ... It is natural that the theoretical and applied works of

> Sergei Natanovich and his text in probability theory have determined to a considerable degree the development of research in probability theory in the USSR.

And then

> 4. A whole series of papers by Sergei Natanovich are connected with the strengthening of Chebyshev's inequality [citing papers from 1918, 1924, and 1937] and the calculation of the error in the Laplace formula ...

I took the quotes from the SIAM translation of Volume V number 2 of the Russian original.

The proof of the Bernstein inequality given by Uspensky (1937, pages 204–206) used the MGF method. He prefaced the "Indication of the Proof" by the remark "S. Bernstein has shown that Tchebycheff's inequality can be considerably improved". I thank Elena Khusainova, who translated four pages from Bernstein's probability book for me, clearly showing that Uspensky was following that book, which appeared in his list of references (page 207).

Now for the view from the West. Hoeffding (1963, page 14) gave Bernstein credit for the MGF approach: "The method employed to derive the inequalities, which has often been used (apparently first by S. N, Bernstein),...". Hoeffding (1963, page 15-16) also commented that Chernoff (1952) had already used the Binomial case as one of his examples. See also the comments by Bennett (1962, page 35): "Bernstein's original work was published in Russian, and appears to be unobtainable. It is reported—indirectly—by [Craig, 1933] ...and by [Uspensky (1937, pages 204–206)] who indicates the proof in a series of exercises. The inequality is mentioned or quoted without proof by ... Apart from these brief references, Bernstein's inequality seems to have escaped notice in the English-speaking world." It is also interesting to note Craig's comment (on his page 94) "Another interesting and important attempt in this direction due to S. N. Bernstein seems to have generally escaped attention in the English-speaking world, at least, since it has been published only in Russian", with the footnote "Bernstein, S., Theory of Probability, (Moscow, 1927), pp. 159-165. The present account of this work of Bernstein is taken from a lecture of Professor J. V. Uspensky." Craig also mentioned that his paper was written while he was at Stanford University, where Uspensky was a mathematics faculty member, until his death in 1947. At least some at Stanford had been aware of Bernstein's contributions.

Inequality <12> for $\mathcal{R}(x) = \bar{\Phi}(x)/\phi(x)$ is classical. It corresponds to the first two terms in the asymptotic expansion derived in Problem [1]. The analogous result for the error function (that is, $\int_0^x e^{-t^2} dt$) together with a continued fraction expansion was given by Laplace in his Celestial Mechanics, reprinted 1805 in Volume IV, Book X, Chapter 1, §5 of his collected works (pages 489–492 in the Bowditch translation).

The ratio $\mathcal{R}(x)$ is often called the Mills ratio, because it was discussed by Mills (1926). Actually Mills was just constructing a table of $\mathcal{R}$, using earlier tables for the normal distribution function and numerical methods proposed by other authors. There has been a long history of authors deriving upper and lower bounds for $\mathcal{R}$, such as the upper bound from Problem [5] and the lower bound from Problem [6]. See Baricz (2008) or Gasull and Utzet (2014) for recent examples, which include some history. As far as I can tell, none of these bounds is sharp enough to compete with the usual numerical methods (Press et al., 1987, Section 6.2) for calculating various tail probabilities. I assume that modern computing power makes printed tables rather irrelevant these days. Nevertheless, some of the host of papers about M do contain interesting theoretical ideas.

The subGamma idea is clearly present in the derivation of the Bernstein inequality from the Bernstein moment assumption (see Section 8.3) but, to my knowledge, Boucheron, Lugosi, and Massart (2013, page 28) were the first to anoint it as a general concept. It seems that the neat trick <25> with the square roots was first noted by Birgé and Massart (1998, Section 7.6), although I would not be surprised if someone else had also noticed the same facts during the long history of the Bernstein inequality. Nevertheless, there is no doubting that the research group associated with Lucien Birgé and Pascal Massart, in Paris, contributed many new and significant ideas.

Okamoto (1959) stated the Binomial tail bounds <29> and its analog for the lower tail in a slightly different form, with the comment that "We shall state two Lemmas the first of which is a corollary of a theorem given by [Chernoff 1952, Theorem 1]". He then derived several more attractive bounds that could be derived from the MGF bound. He omitted the Calculus (which I provided in Section 3.7) for the MGF bound; he only gave the details for the weaker upper bounds. He also commented that his proof simplified a "tedious, although elementary" calculation by Uspensky (1937, page 102). It seems strange to me that Okamoto did not also cite Uspensky (1937, page 204).

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