Chapter 6

Martingale et al.

SECTION 1 gives some examples of martingales, submartingales, and supermartingales.

SECTION 2 introduces stopping times and the sigma-fields corresponding to "information available at a random time." A most important Stopping Time Lemma is proved, extending the martingale properties to processes evaluted at stopping times.

SECTION 3 shows that positive supermartingales converge almost surely.

SECTION 4 presents a condition under which a submartingale can be written as a difference between a positive martingale and a positive supermartingale (the Krickeberg decomposition). A limit theorem for submartingales then follows.

*SECTION *5 proves the Krickeberg decomposition.*

*SECTION *6 defines uniform integrability and shows how uniformly integrable martingales are particularly well behaved.*

*SECTION *7 show that martingale theory works just as well when time is reversed.*

*SECTION *8 uses reverse martingale theory to study exchangeable probability measures on infinite product spaces. The de Finetti representation and the Hewitt-Savage zero-one law are proved.*

1. What are they?

The theory of martingales (and submartingales and supermartingales and other related concepts) has had a profound effect on modern probability theory. Whole branches of probability, such as stochastic calculus, rest on martingale foundations. The theory is elegant and powerful: amazing consequences flow from an innocuous assumption regarding conditional expectations. Every serious user of probability needs to know at least the rudiments of martingale theory.

A little notation goes a long way in martingale theory. A fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$ sits in the background. The key new ingredients are:

(i) a subset T of the extended real line $\overline{\mathbb{R}}$;

(ii) a ***filtration*** $\{\mathcal{F}_t : t \in T\}$, that is, a collection of sub-sigma-fields of $\mathcal{F}$ for which $\mathcal{F}_s \subseteq \mathcal{F}_t$ if $s < t$;

(iii) a family of integrable random variables $\{X_t : t \in T\}$ ***adapted*** to the filtration, that is, X_t is $\mathcal{F}_t$-measurable for each t in T.