Chapter 7

Convergence in distribution

SECTION 1 defines the concepts of weak convergence for sequences of probability measures on a metric space, and of convergence in distribution for sequences of random elements of a metric space and derives some of their consequences. Several equivalent definitions for weak convergence are noted.

SECTION 2 establishes several more equivalences for weak convergence of probability measures on the real line, then derives some central limit theorems for sums of independent random variables by means of Lindeberg's substitution method.

SECTION 3 explains why the multivariate analogs of the methods from Section 2 are not often explicitly applied.

SECTION 4 develops the calculus of stochastic order symbols.

*SECTION *5 derives conditions under which sequences of probability measures have weakly convergent subsequences.*

1. Definition and consequences

Roughly speaking, central limit theorems give conditions under which sums of random variable have approximate normal distributions. For example:

> If $\xi_1, \ldots, \xi_n$ are independent random variables with $\mathbb{P}\xi_i = 0$ for each i and $\sum_i \operatorname{var}(\xi_i) = 1$, and if none of the ξ_i makes too large a contribution to their sum, then $\sum_i \xi_i$ is approximately $N(0,1)$ distributed.

The traditional way to formalize approximate normality requires, for each real x, that $\mathbb{P}\{\sum_i \xi_i \le x\} \approx \mathbb{P}\{Z \le x\}$ where Z has a $N(0,1)$ distribution. Of course the variable Z is used just as a convenient way to describe a calculation with the $N(0,1)$ probability measure; Z could be replaced by any other random variable with the same distribution. The assertion does not mean that $\sum_i \xi_i \approx Z$, as functions defined on a common probability space. Indeed, the Z need not even live on the same space as the $\{\xi_i\}$. We could remove the temptation to misinterpret the approximation by instead writing $\mathbb{P}\{\sum_i \xi_i \le x\} \approx P(-\infty, x]$ where P denotes the $N(0,1)$ probability measure.

Assertions about approximate distributions of random variables are usually expresssed as limit theorems. For example, the sum could be treated as one of a sequence of such sums, with the approximation interpreted as an assertion of convergence to a limit. We thereby avoid all sorts of messy details about the size of