Chapter 2

A modicum of measure theory

SECTION 1 defines measures and sigma-fields.
SECTION 2 defines measurable functions.
SECTION 3 defines the integral with respect to a measure as a linear functional on a cone of measurable functions. The definition sidesteps the details of the construction of integrals from measures.
*SECTION *4 constructs integrals of nonnegative measurable functions with respect to a countably additive measure.*
SECTION 5 establishes the Dominated Convergence theorem, the Swiss Army knife of measure theoretic probability.
SECTION 6 collects together a number of simple facts related to sets of measure zero.
*SECTION *7 presents a few facts about spaces of functions with integrable pth powers, with emphasis on the case p=2, which defines a Hilbert space.*
SECTION 8 defines uniform integrability, a condition slightly weaker than domination. Convergence in $\mathcal{L}^1$ is characterized as convergence in probability plus uniform integrability.
SECTION 9 defines the image measure, which includes the concept of the distribution of a random variable as a special case.
SECTION 10 explains how generating class arguments, for classes of sets, make measure theory easy.
*SECTION *11 extends generating class arguments to classes of functions.*

1. Measures and sigma-fields

As promised in Chapter 1, we begin with measures as set functions, then work quickly towards the interpretation of integrals as linear functionals. Once we are past the purely set-theoretic preliminaries, I will start using the de Finetti notation (Section 1.4) in earnest, writing the same symbol for a set and its indicator function.

Our starting point is a ***measure space***: a triple $(\mathcal{X}, \mathcal{A}, \mu)$, with $\mathcal{X}$ a set, $\mathcal{A}$ a class of subsets of $\mathcal{X}$, and μ a function that attaches a nonnegative number (possibly $+\infty$) to each set in $\mathcal{A}$. The class $\mathcal{A}$ and the set function μ are required to have properties that facilitate calculations involving limits along sequences.