Chapter 8

Fourier transforms

SECTION 1 presents a few of the basic properties of Fourier transforms that make them such a valuable tool of probability theory.

SECTION 2 exploits a mysterious coincidence, involving the Fourier transform and the density function of the normal distribution, to establish inversion formulas for recovering distributions from Fourier transforms.

*SECTION *3 explains why the coincidence from Section 2 is not really so mysterious.*

SECTION 4 shows that the inversion formula from Section 2 has a continuity property, which explains why pointwise convergence of Fourier transforms implies convergence in distribution.

*SECTION *5 establishes a central limit theorem for triangular arrays of martingale differences.*

SECTION 6 extends the theory to multivariate distributions, pointing out how the calculations reduce to one-dimensional analogs for linear combinations of coordinate variables—the Cramér and Wold device.

*SECTION *7 provides a direct proof (no Fourier theory) of the fact that the family of (one-dimensional) distributions for all linear combinations of a random vector uniquely determines its multivariate distribution.*

*SECTION *8 illustrates the use of complex-variable methods to prove a remarkable property of the normal distribution—the Lévy-Cramér theorem.*

1. Definitions and basic properties

Some probabilistic calculations simplify when reexpressed in terms of suitable transformations, such as the probability generating function (especially for random variables taking only positive integer values), the Laplace transform (especially for random variables taking only nonnegative values), or the moment generating function (for random variables with rapidly decreasing tail probabilities). The Fourier transform shares many of the desirable properties of these transforms without the restrictions on the types of random variable to which it is best applied, but with the slight drawback that we must deal with random variables that can take complex values.

The integral of a complex-valued function, $f := g + ih$, is defined by splitting into real ($\Re f := g$) and imaginary ($\Im f := h$) parts, $\mu f := \mu g + i\mu h$. These integrals inherit linearity and the dominated convergence property from their real-valued