Chapter 9

Brownian motion

SECTION 1 collects together some facts about stochastic processes and the normal distribution, for easier reference.

SECTION 2 defines Brownian motion as a Gaussian process indexed by a subinterval T of the real line. Existence of Brownian motions with and without continuous sample paths is discussed. Wiener measure is defined.

SECTION 3 constructs a Brownian motion with continuous sample paths, using an orthogonal series expansion of square integrable functions.

*SECTION *4 describes some of the finer properties—lack of differentiability, and a modulus of continuity—for Brownian motion sample paths.*

SECTION 5 establishes the strong Markov property for Brownian motion. Roughly speaking, the process starts afresh as a new Brownian motion after stopping times.

*SECTION *6 describes a family of martingales that can be built from a Brownian motion, then establishes Lévy's martingale characterization of Brownian motion with continuous sample paths.*

*SECTION *7 shows how square integrable functions of the whole Brownian motion path can be represented as limits of weighted sums of increments. The result is a thinly disguised version of a remarkable property of the isometric stochastic integral, which is mentioned briefly.*

*SECTION *8 explains how the result from Section 7 is the key to the determination of option prices in a popular model for changes in stock prices.*

1. Prerequisites

Broadly speaking, Brownian motion is to stochastic process theory as the normal distribution is to the theory for real random variables. They both arise as natural limits for sums of small, independent contributions; they both have rescaling and transformation properties that identify them amongst wider classes of possible limits; and they have both been studied in great detail. Every probabilist, and anyone dealing with continuous-time processes, should learn at least a little about Brownian motion, one of the most basic and most useful of all stochastic processes. This Chapter will define the process and explain a few of its properties.

The discussion will draw on a few basic ideas about stochastic processes, and a few facts about the normal distribution, which are summarized in this Section.