Chapter 11

Exponential tails and the law of the iterated logarithm

SECTION 1 introduces the law of the iterated logarithm (LIL) through the technically simplest case: independent standard normal summands.

SECTION 2 extends the results from Section 1 to sums of independent bounded random variables, by means of Bennett's exponential inequality. It is noted that the bounds on the variables could increase slowly without destroying the limit assertion, thereby pointing to the easy (upper) half of Kolmogorov's definitive LIL.

*SECTION *3 derives the very delicate exponential lower bound for bounded summands, needed to prove the companion lower half for Kolmogorov's LIL.*

*SECTION *4 shows how truncation arguments extend Kolmogorov's LIL to the case of independent, identically distributed summands with finite second moments.*

1. LIL for normal summands

Two important ideas run in tandem through this Chapter: the existence of exponential tail bounds for sums of independent random variables, and proofs of the law of the iterated logarithm (LIL) in various contexts. You could read the Chapter as either a study of exponential inequalities, with the LIL as a guiding application, or as a study of the LIL, with the exponential inequalities as the main technical tool.

The LIL's will all refer to partial sums $S_n := X_1 + \ldots + X_n$ for sequences of independent random variables $\{X_i\}$ with $\mathbb{P}X_i = 0$ and $\text{var}(X_i) := \sigma_i^2 < \infty$, for each i. The words *iterated logarithm* refer to the role played by function $L(x) := \sqrt{2x \log\log x}$. To avoid minor inconveniences (such as having to exclude cases involving logarithms or square roots of negative numbers), I arbitrarily define $L(x)$ as 1 for $x < e^e \approx 15.15$. Under various assumptions, we will be able to prove, with $V_n := \text{var}(S_n)$, that

<1> $$\limsup_{n\to\infty} S_n/L(V_n) = 1 \qquad \text{almost surely,}$$

together with analogous assertions about the lim inf and the almost sure behavior of the sequence $\{S_n/L(V_n)\}$. Equality <1> breaks naturally into a pair of assertions,

<2> $$\limsup_{n\to\infty} S_n/L(V_n) \le 1 \qquad \text{and} \qquad \limsup_{n\to\infty} S_n/L(V_n) \ge 1 \qquad \text{a.s.,}$$