

Chapter 12

Multivariate normal distributions

SECTION 1 explains why you will not learn from this Chapter everything there is to know about the multivariate normal distribution.

SECTION 2 introduces Fernique's inequality. As illustration, Sudakov's lower bound for the expected value of a maximum of correlated normals is derived.

*SECTION *3 proves Fernique's inequality.*

SECTION 4 introduces the Gaussian isoperimetric inequality. As an application, Borell's tail bound for the distribution of the maximum of correlated normals is derived.

*SECTION *5 proves the Gaussian isoperimetric inequality.*

1. Introduction

Of all the probability distributions on multidimensional Euclidean spaces the multivariate normal is the most studied and, in many ways, the most tractable. In years past, the statistical subject known as “Multivariate Analysis” was almost entirely devoted to the study of the multivariate normal. The literature on Gaussian processes—stochastic processes whose finite dimensional distributions are all multivariate normal—is vast. It is important to know a little about the multivariate normal.

As you saw in Section 8.6, the multivariate normal is uniquely determined by its vector of means and its matrix of covariances. In principle, everything that one might want to know about the distribution can be determined by calculation of means and covariances, but in practice it is not completely straightforward. In this Chapter you will see two elegant examples of what can be achieved: Fernique's (1975) inequality, which deduces important information about the spread in a multivariate normal distribution from its covariances; and Borell's (1975) Gaussian isoperimetric inequality, with a proof due to Ehrhard (1983a, 1983b). Both results are proved by careful Calculus.

The Chapter provides only a very brief glimpse of Multivariate Analysis and the theory of Gaussian processes, two topics that are covered in great detail in many specialized texts. I have chosen merely to present examples that give the flavor of some of the more modern theory. Both the Fernique and Borell inequalities have found numerous applications in the recent research literature.