Appendix A

Measures and integrals

SECTION 1 introduces a method for constructing a measure by inner approximation, starting from a set function defined on a lattice of sets.

SECTION 2 defines a "tightness" property, which ensures that a set function has an extension to a finitely additive measure on a field determined by the class of approximating sets.

SECTION 3 defines a "sigma-smoothness" property, which ensures that a tight set function has an extension to a countably additive measure on a sigma-field.

SECTION 4 shows how to extend a tight, sigma-smooth set function from a lattice to its closure under countable intersections.

SECTION 5 constructs Lebesgue measure on Euclidean space.

SECTION 6 proves a general form of the Riesz representation theorem, which expresses linear functionals on cones of functions as integrals with respect to countably additive measures.

1. Measures and inner measure

Recall the definition of a ***countably additive measure*** on ***sigma-field***. A sigma-field $\mathcal{A}$ on a set $\mathcal{X}$ is a class of subsets of $\mathcal{X}$ with the following properties.

(SF_1) *The empty set $\emptyset$ and the whole space $\mathcal{X}$ both belong to $\mathcal{A}$.*

(SF_2) *If A belongs to $\mathcal{A}$ then so does its complement A^c.*

(SF_3) *For countable $\{A_i : i \in \mathbb{N}\} \subseteq \mathcal{A}$, both $\cup_i A_i$ and $\cap_i A_i$ are also in $\mathcal{A}$.*

A function μ defined on the sigma-field $\mathcal{A}$ is called a countably additive (nonnegative) measure if it has the following properties.

(M_1) $\mu\emptyset = 0 \le \mu A \le \infty$ *for each A in $\mathcal{A}$.*

(M_2) $\mu\left(\cup_i A_i\right) = \sum_i \mu A_i$ *for sequences $\{A_i : i \in \mathbb{N}\}$ of pairwise disjoint sets from $\mathcal{A}$.*

If property SF_3 is weakened to require stability only under finite unions and intersections, the class is called a ***field***. If property M_2 is weakened to hold only for disjoint unions of finitely many sets from $\mathcal{A}$, the set function is called a ***finitely additive measure***.

Where do measures come from? Typically one starts from a nonnegative real-valued set-function μ defined on a small class of sets $\mathcal{K}_0$, then extends to a sigma-field $\mathcal{A}$ containing $\mathcal{K}_0$. One must at least assume "measure-like" properties for μ on $\mathcal{K}_0$ if such an extension is to be possible. At a bare minimum,