Appendix B

Hilbert spaces

SECTION 1 defines Hilbert space and presents two basic inequalities.
SECTION 2 establishes the existence of orthogonal projections onto closed subspaces of a Hilbert space.
SECTION 3 defines orthonormal bases of Hilbert spaces. Vectors in the space have representations as infinite linear combinations (convergent series) of basis vectors.
SECTION 4 shows how to construct a random process from an orthonormal sequence of random variables and an orthonormal basis.

1. Definitions

Hilbert space is an infinite dimensional generalization of ordinary Euclidean space. Arguments involving Hilbert spaces look similar to their analogs for Euclidean space, with the addition of occasional precautions against possible difficulties with infinite dimensionality.

<1> **Definition.** *A Hilbert space is a vector space* $\mathcal{H}$ *equipped with an* ***inner product*** $\langle \cdot, \cdot \rangle$ *(a map from* $\mathcal{H} \otimes \mathcal{H}$ *into* $\mathbb{R}$*) which satisfies the following requirements.*

(a) $\langle \alpha f + \beta g, h \rangle = \alpha \langle f, h \rangle + \beta \langle g, h \rangle$ *for all real* α, β *all* f, g, h *in* $\mathcal{H}$.

(b) $\langle f, g \rangle = \langle g, f \rangle$ *for all* f, g *in* $\mathcal{H}$.

(c) $\langle f, f \rangle \geq 0$ *with equality if and only if* $f = 0$.

(d) $\mathcal{H}$ *is complete for the norm defined by* $\| f \| := \sqrt{\langle f, f \rangle}$. *That is, if* $\{f_n\}$ *is a* ***Cauchy sequence*** *in* $\mathcal{H}$, *meaning* $\| f_n - f_m \| \to 0$ *as* $\min(m, n) \to \infty$, *then there exists an* f *in* $\mathcal{H}$ *for which* $\| f_n - f \| \to 0$.

Two elements f and g of $\mathcal{H}$ are said to be ***orthogonal***, written $f \perp g$, if $\langle f, g \rangle = 0$. An element f is said to be orthogonal to a subset G of $\mathcal{H}$, written $f \perp G$, if $f \perp g$ for every g in G.

The prime examples of Hilbert spaces are ordinary Euclidean space and $L^2(\mu)$, the set of equivalence classes of measurable real-valued functions whose squares are μ-integrable, for a fixed measure μ. See Section 2.7 for discussion of why we need to work with μ-equivalence classes to get property (c).

Hilbert space shares several properties with ordinary Euclidean space.