Appendix C

Convexity

SECTION 1 defines convex sets and functions.
SECTION 2 shows that convex functions defined on subintervals of the real line have left- and right-hand derivatives everywhere.
SECTION 3 shows that convex functions on the real line can be recovered as integrals of their one-sided derivatives.
SECTION 4 shows that convex subsets of Euclidean spaces have nonempty relative interiors.
SECTION 5 derives various facts about separation of convex sets by linear functions.

1. Convex sets and functions

A subset C of a vector space is said to be convex if it contains all the line segments joining pairs of its points, that is,

$$\alpha x_1 + (1-\alpha)x_2 \in C \qquad \text{for all } x_1, x_2 \in C \text{ and all } 0 < \alpha < 1.$$

A real-valued function f defined on a convex subset C (of a vector space $\mathcal{V}$) is said to be convex if

$$f(\alpha x_1 + (1-\alpha)x_2) \le \alpha f(x_1) + (1-\alpha) f(x_2) \qquad \text{for all } x_1, x_2 \in C \text{ and } 0 < \alpha < 1.$$

Equivalently, the ***epigraph*** of the function,

$$\text{epi}(f) := \{(x,t) \in C \times \mathbb{R} : t \ge f(x)\},$$

is a convex subset of $C \times \mathbb{R}$. Some authors (such as Rockafellar 1970) define $f(x)$ to equal $+\infty$ for $x \in \mathcal{V}\backslash C$, so that the function is convex on the whole of $\mathcal{V}$, and $\text{epi}(f)$ is a convex subset of $\mathcal{V} \times \mathbb{R}$.

This Appendix will establish several facts about convex functions and sets, mostly for Euclidean spaces. In particular, the facts include the following results as special cases.

(i) For a convex function f defined at least on an open interval of the real line (possibly the whole real line), there exists a countable collection of linear functions for which $f(x) = \sup_{i \in \mathbb{N}} (\alpha_i + \beta_i x)$ on that interval.

(ii) If a real-valued function f has an increasing, real-valued right-hand derivative at each point of an open interval, then f is convex on that interval. In particular, if f is twice differentiable, with $f'' \ge 0$, then f is convex.