Appendix D

Binomial and normal distributions

SECTION 1 establishes some useful bounds for the tails of the normal distribution, then uses them to derive the perturbation inequalities needed for the proof of the main result, in Section 2

SECTION 2 describes a very precise approximation to symmetric Binomial tail probabilities via the tails of the standard normal distribution. The approximation implies existence of a very tight coupling between the Binomial and its approximating normal—the key to the KMT coupling (Chapter 10) between the empirical process and a Brownian Bridge.

SECTION 3 proves the results described in Section 2.

1. Tails of the normal distributions

The $N(0,1)$ distribution on the real line has density function $\phi(x) := \exp(-x^2/2)/\sqrt{2\pi}$ with respect to Lebesgue measure. For many limit theorems and inequalities it is only the rate of decrease of the tail probability $\bar{\Phi}(x) := \mathbb{P}\{N(0,1) > x\}$ that matters. The simplest approximation,

<1> $$\left(\frac{1}{x} - \frac{1}{x^3}\right)\phi(x) < \bar{\Phi}(x) < \frac{1}{x}\phi(x) \qquad \text{for } x > 0,$$

follows (compare with Feller 1968, Section VII.1 and Problem 7.1) by integrating from x to ∞ across the trivial inequalities

$$\left(1 - \frac{3}{t^4}\right)\phi(t) < \phi(t) < \left(1 + \frac{1}{t^2}\right)\phi(t) \qquad \text{for } t > 0.$$

Less precisely,

$$\bar{\Phi}(x) = \frac{\phi(x)}{x}\left(1 - O(x^{-2}\right) \qquad \text{as } x \to \infty.$$

When x is close to zero there are a better bounds, such as

<2> $$\bar{\Phi}(x) \le \frac{1}{2}\exp\left(-x^2/2\right) \qquad \text{for } x \ge 0,$$

an inequality that will follow from properties of the function $\rho(x) := \phi(x)/\bar{\Phi}(x)$, defined for all real x.